A new quantization condition for parity-violating three-dimensional gravity

Timothy Blais

Physics Department
Master of Science

McGill University, Montreal

April 12, 2013
A thesis submitted to McGill University in partial fulfilment of the requirements for the degree of Master of Science.

Acknowledgements

Obviously I owe a huge debt of gratitude to my supervisor Alex Maloney for his time, his guidance and especially his willingness to patiently explain the obvious. Without his ideas and suggestions this work would never have begun, let alone reached a result. Thanks to Johannes Walcher for helpful conversations near the beginning of this project, and to Kefeng Liu and Hao Xu for graciously allowing us to use their computer code. Many thanks to Alexandre Belin for being my research partner for much of the last two years, to Alejandra Castro and Josh Lapan for their invaluable string theory courses, and to Arnaud Lepage-Jutier for many helpful explanations and also for proofreading my French. Thanks also to NSERC for their financial support.

I would like to express my deep gratitude to all my friends and family for being so understanding of my complete absence from their lives during much of this process, and a special thanks to Natalie Richards and Laura Griffin for their constant efforts to keep me sane. Most importantly I must thank God, the foundation of all order, to whom I owe both the theories I study and the brain to make sense of them, and without whom I would have given up on this project long ago.

Abstract

(2+1)-dimensional gravity with a negative cosmological constant is a topological theory with no local degrees of freedom. When confined to compact universes which are topologically genus g Riemann surfaces times time, its classical phase space is the cotangent bundle of the moduli space of Riemann surfaces. We consider the quantization of moduli space itself, emerging as the zero-momentum slice of this phase space. When a parity-violating Chern-Simons term is added to the gravitational action, a nontrivial symplectic form is induced on this slice which is a multiple of the Weil-Petersson Kähler form. By demanding that this symplectic form integrate to $2\pi\hbar$ times an integer on every nontrivial two-cycle in moduli space—which is a necessary condition for the system to be quantizable—we find a new quantization condition on the Chern-Simons coupling k'. Our result strongly suggests that k' must be an integer multiple of 24 in order to define a self-consistent theory of quantum gravity.

Abrégé

La gravitation en 2+1 dimensions avec une constante cosmologique négative est une théorie topologique, sans degrés de liberté locaux. Lorsqu'elle est limitée à des univers compacts qui sont topologiquement un produit direct des surfaces de Riemann au genre g et du temps, son espace de phase classique est le fibré cotangent de l'espace des modules des surfaces de Riemann. Nous considérons la quantification de l'espace des modules lui-même, qui se produit comme la tranche zéro dynamique de cet espace de phase. Quand on ajoute un terme Chern-Simons, qui brise la parité, à l'action gravitationelle, une forme symplectique non triviale est induite sur cette tranche qui est un multiple de la forme Kähler de Weil-Petersson. En exigeant que cette forme symplectique doit intégrer à $2\pi\hbar$ multiplié par un entier sur tous les cycles non triviaux de degré 2 dans l'espace des modules—condition nécessaire pour faire la quantification du système—nous trouvons une nouvelle condition de quantification pour k', le couplage Chern-Simons. Notre résultat suggère fortement que k' doit être un multiple entier de 24, afin de définir une théorie cohérente de la gravitation quantique.

Contents

1 Introduction **1**

1.1 Content of this work 3

2 3d gravity **7**

2.1 3d Einstein gravity 7

2.2 The ADM formalism 11

2.3 3d gravity as a Chern-Simons theory 14

2.4 Parity-violating Chern-Simons gravity 21

3 Statement of the Problem **23**

3.1 A minisuperspace model 23

3.2 Particle in a uniform magnetic field 26

3.2.1 The free particle 26

3.2.2 Addition of a uniform magnetic field 27

4 Geometric quantization **32**

4.1 Phase space as a symplectic manifold 33

4.2 Prequantization 36

4.3 Polarization 41

4.3.1 Real polarization 42

4.3.2 Kähler polarization 44

4.4 Geometric quantization of 3d gravity 48

5 The geometry of moduli space **52**

5.1 Teichmüller space 53

5.2 The Weil-Petersson Form 57

5.3 Moduli space 60

5.4 Volumes of two-dimensional moduli spaces 61

5.5 Cohomology of the Weil-Petersson form 64

6 Results and Outlook **68**

6.1 The quantization condition 68

6.2 Semiclassical behaviour of the Hilbert space 69

6.3 Outlook 74

Chapter 1

Introduction

Despite both being nearly a hundred years old, the theories of quantum mechanics and general relativity have yet to be combined in a completely satisfactory way. There are several reasons for this. The first is that when we attempt to treat general relativity perturbatively as a quantum field theory, we immediately run into intractable infinities. Fundamentally this is because general relativity, whose coupling constant $G^{1/2}$ has units of inverse energy, is a nonrenormalizable theory: it requires a UV completion to properly understand its behaviour at high energies and short distance scales. The search for this UV completion has given rise to several theories, the most popular being string theory, which purport to describe this short-distance physics in such a way that quantum mechanics can properly be applied to the theory. While these programs have yet to make contact with experimental reality, they have led to rich mathematical theories and spawned many influential ideas, such as compact extra dimensions, brane worlds, spacetime holography and noncommutative geometry, which continue to shape the way we think about gravity and the nature of spacetime.

There is, however, more to understanding quantum gravity than simply finding the UV completion of a particular nonrenormalizable field theory. Rather, since general relativity is the study of a dynamical spacetime, quantum gravity must describe how spacetime itself emerges from a quantum theory. This is a much stranger puzzle than a quantum field theory, which takes the background spacetime as a given. Furthermore, general relativity, being diffeomorphism invariant, seems to require a diffeomorphism-invariant quantum theory, which does away with the usual locality of quantum field theory as well as any notion of a canonical time-slice on which to normalize quantum probabilities. We would like to have a theory that allows us to test out these oddities of quantum gravity without being hampered by the nonrenormalizability of the theory.

(2+1)-dimensional gravity provides us with such a theory. In universes with this dimensionality, one less than our own universe (macroscopically, at least), the local spacetime geometry is completely fixed by Einstein's equations of motion and there are no local degrees of freedom. This results in a trivial theory on topologically trivial manifolds, but when spacetime is nontrivially identified there are a finite number of parameters that define its topology, and these become our only dynamical degrees of freedom. Quantizing this theory therefore becomes a quantum mechanics problem rather than a quantum field theory, and renormalizability becomes a non-issue. This leaves us free to worry about other questions in quantum gravity, such as: What is a state in a quantum gravity? What determines the initial state of a big bang universe? What is the natural topology of spacetime, and can it change? And are there inequivalent ways to quantize a single classical gravity theory?

The classical theory of 3d gravity dates back to Staruszkiewicz [1], who first considered the problem of point particles in 3d gravity in 1963. The quantum theory first came to real fruition with the advent of the Chern-Simons formulation of 3d gravity, first discovered by Achúcarro and Townsend in 1986 [2] and rediscovered by Witten [3] in 1988, which allows the space of classical solutions to be quantized as a topological field theory. More recently, 3d gravity has proven quite useful in the realm of the AdS/CFT correspondence, providing easy checks of black hole entropy as a counting of holographic states [4] and of holographic entanglement entropy as the area of minimal surfaces [5].

Here we will consider pure 3d gravity, without any matter content, as it applies to compactified big-bang/big-crunch universes. We will work with an extension of classical Einstein gravity which includes a parity-violating term. This term introduces a nontrivial commutator on the 'ground state' space of the theory: the subset of classical solutions whose dynamics are governed by a single time-dependent scale factor. This ground state space is the moduli space of Riemann surfaces, and is an orbifold due to the large-diffeomorphism invariance of gravity. This means that our phase space is topologically nontrivial, and the requirement that wave functions on this space must be single-valued may lead to a quantization condition on the parameters of the theory. We will attempt to find this condition, and determine whether it changes with the topology of the spacetime being considered. In the cases where our phase space *is* quantizable, will examine the resulting theory and attempt to find its Hilbert space using the methods of geometric quantization.

1.1 Content of this work

This work proceeds as follows. In Chapter 2 we provide an overview of the solutions of 3d gravity and present two different frameworks for describing its dynamics: the ADM formalism and the Chern-Simons formulation. These frameworks complement each other and together allow us to get an intuition for the phase space of the theory. Ultimately though, we will be more interested in the second one, as it is the one most easily extended to the parity-violating theory that we wish to consider. We will see that the classical phase space remains the same whether or not a parity-violating term is present in the action, and can be described as the tangent bundle $T^*\mathcal{M}_g$ of the moduli space of Riemann surfaces. However, the parity-violating term does change the Poisson brackets of the theory, and leads to an inequivalent quantum mechanics.

In Chapter 3 we lay out the problem we wish to consider: the quantization of moduli space $\mathcal{M}_g$ as the 'zero-momentum' slice of our full phase space. In the general case where the coefficient k' of the parity-violating term is nonzero, there is a nontrivial quantum mechanics on $\mathcal{M}_g$ with a commutator proportional to $1/k'$. The nontrivial topology of $\mathcal{M}_g$, and in particular its orbifold structure, leads us to expect that single-valuedness of wave functions around nontrivial cycles will lead to a quantization condition on k'; our goal is to find this condition. We provide a simple motivating example from quantum particle theory which illustrates the drastic effects of a parity-violating term on a quantum theory, and how such a theory still retains the important properties of its ground-state Hilbert space when confined to the zero-momentum slice of its phase space.

Chapter 4 is devoted to an exposition of the framework and relevant aspects of geometric quantization (GQ). GQ is a quantization scheme based on symplectic geometry that allows one to systematically build a Hilbert space out of a phase space and a commutator, represented as a symplectic manifold M and a closed nondegenerate two-form ω called the symplectic form. We show how wave functions in the theory are constructed as global sections of a certain complex line bundle, the 'prequantum line bundle', which must have a compatible connection with curvature equal to $(1/\hbar)\omega$. The question of single-valuedness of the wave function in this case boils down to the question of whether or not such a line bundle exists. A necessary and sufficient condition for this is that $(1/2\pi\hbar)\omega$ must be in integer cohomology. In our case $(1/2\pi\hbar)\omega = (k'/4\pi^2)\omega_{WP}$ where ω_{WP} is the well-studied Weil-Petersson Kähler form on moduli space, so the quantization condition on our theory becomes the condition that this form be in integer cohomology. We briefly present the notion of polarizations in GQ, and show that, if quantizable, the Hilbert space of our reduced theory is the space of holomorphic sections of the prequantum line bundle.

In Chapter 5 we delve into the geometry of the general moduli space $\mathcal{M}_{g,n}$ of a genus g Riemann surfaces with n marked points. We show how its universal covering space $\mathcal{T}_{g,n}$, called Teichmüller space, can be constructed in Fenchel-Nielsen coordinates by considering the decomposition of the surface into 'pairs of pants.' The mapping class group Γ, the group of large diffeomorphisms on the surface, is generated by Dehn twists: operations which cut along a closed geodesic and twist by 2π. Moduli space is then constructed as $\mathcal{M}_{g,n} = \mathcal{T}_{g,n}/\Gamma$. We show how the Weil-Petersson form arises as the natural Kähler form in these coordinates, and how it survives as a Kähler form under the quotient of the mapping class group. We then derive the cohomology of ω_{WP} on $\mathcal{M}_{g,n}$ following the analysis by Wolpert [6], and conclude that $(1/4\pi^2)\omega_{WP}$ is in $\frac{1}{24}$ integer cohomology.

Finally in Chapter 6 we present our main result: a new quantization condition on the coupling constant k'. We show that k' must be a multiple of 24 when we take our phase space to be the Delign-Mumford compactification $\overline{\mathcal{M}}_g$, and that this remains a sufficient condition for the quantizability of the uncompactified space although we cannot rule out the possibility that some rational fraction of this quantity is also allowed. We show that the quantization condition for our reduced model, though found by confining to the zero-momentum slice of the phase space, provides a necessary condition for the quantization of the full theory. We discuss the semiclassical dimensionality of the quantum mechanical Hilbert space, using results from GQ and a conjecture by Zograf, and find evidence of a divergence that suggests that a 'typical' state in the theory is an infinite-genus handle body: a sort of spacetime foam. We perform a preliminary examination of how this divergence is affected by the addition of a length scale into the problem, and attempt to relate this to the finite number of microstates behind a BTZ black hole event horizion. We conclude by suggesting some directions for future research.

Chapter 2

3d gravity

To start off, we present an exposition of gravity in 2+1 dimensions. We will be concerned only with pure gravity, with an action dependent only on the geometry of the space with no external fields. We will look at two theories of gravity: classical Einstein gravity, as well as an extension which includes an 'exotic' parity-violating term. Our goal in this section will be to describe the phase space of all solutions to these theories whose spatial slice is a compact Riemann surface, and to analyze how such a phase space might be quantized.

Many of the results in section 2.1 are standard results in general relativity, and derivations can be found in textbooks such as Carroll [7]. Standard results which are specific to 3d general relativity, as well as those on the first-order formalism in section 2.3 can be found in such works as the review and book by Carlip [8][9].

2.1 3d Einstein gravity

Let us consider a general d-dimensional spacetime manifold M, with $d-1$ spatial dimensions and a single time dimension, endowed with a Lorentzian metric $g_{\mu\nu}$. We will work in $(-,+,\ldots,+)$ metric signature. In any theory of pure gravity, the metric is the only dynamical variable in the equations of motion, so any action we can construct must be a scalar depending solely on the metric. We require that this scalar be diffeomorphism-invariant, so that the theory depends only on the geometry, not the chosen coordinate system.

In Einstein gravity (with a cosmological constant) our starting point is the Einstein-Hilbert action:

$$I = \frac{1}{16\pi G}\int_M d^3x\sqrt{-g}(R-2\Lambda) \tag{2.1}$$

where R is the Ricci scalar of the metric and Λ is the cosmological constant. Taking the variation of the action with respect to the metric yields Einstein's equations

$$R_{\mu\nu} = \frac{2\Lambda}{d-2}g_{\mu\nu} \tag{2.2}$$

where $R_{\mu\nu}$ is the Ricci scalar. A manifold whose metric satisfies these equations of motion is called an Einstein manifold. In 3+1 dimensions these equations of motion do not entirely fix the local geometry of the space, and at each point there are two local degrees of freedom: the two polarizations of gravitational waves (see, e.g. Carroll [7]). 3+1 dimensional gravity must therefore be quantized as a quantum field theory, with an infinite-dimensional phase space of classical solutions. Unfortunately the Einstein-Hilbert action turns out to be nonrenormalizable, and so attempting to quantize the full theory leads to intractable infinities that prevent us from being able to say anything useful.

In 2+1 dimensions the situation is much changed. Here the local geometry is uniquely fixed by Einstein's equations, due to the fact that the full curvature tensor of the metric can be written entirely in terms of the Ricci tensor:

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R \tag{2.3}$$

Thus, every space-time manifold that is a solution to the (2+1)-dimensional Einstein equations of motion is a simple constant-curvature space. So even though the theory is technically still nonrenormalizable (by power-counting of the coupling constant), there are no propagating modes and there is no need to add extra counterterms to the action when quantizing. We only have to work with renormalized coupling constants G and Λ for the entire theory to be well-behaved.

If we confine ourselves to simply connected manifolds, our only possible classical solutions are the (2+1)-dimensional maximally symmetric spaces: Minkowski space (denoted $\mathbb{R}^{1,2}$) for $\Lambda = 0$, de Sitter (dS_3) space for $\Lambda > 0$ and Anti-de Sitter (AdS_3) space for $\Lambda < 0$. These are not the only Einstein manifolds in 2+1 dimensions, however. We can construct others by quotienting the maximally symmetric spacetimes by discrete subgroups of their symmetry groups: the Poincaré group $ISO(2,1)$ for Minkowski space, $SO(3,1)$ for dS_3 and $SO(2,2)$ for AdS_3. The discrete subgroup then becomes the fundamental group of a topologically nontrivial constant-curvature space. In general these spaces are difficult to classify and may include closed time-like curves.

In this work we will be interested in the case where the manifold is topologically a compact surface times time; i.e. $M \approx \mathbb{R} \times \Sigma$. We will be interested in finding the phase space of all classical solutions of this type, so that later we can quantize this phase space. In this case the fundamental group of the manifold $\pi_1(M)$ is isomorphic to that of its spatial component, $\pi_1(\Sigma)$. Let us take Σ to be a genus g handle body. Then its fundamental group π_1 is generated by $2g$ closed curves in pairs (A_i, B_i), along with the one nontrivial group multiplication:

$$A_1 B_1 A_1^{-1} B_1^{-1} A_2 B_2 A_2^{-1} B_2^{-1} \dots A_g B_g A_g^{-1} B_g^{-1} = 1 \tag{2.4}$$

To obtain a compact constant-curvature space, we choose a map from $\pi_1(\Sigma)$ onto a discrete subgroup of G, the symmetry group of the relevant universal covering space. Quotienting the covering space by this discrete group then produces a space with topology $\mathbb{R} \times \Sigma$, and specifies a constant-curvature metric on it. Two such maps will result in the same metric if they differ by overall conjugation by an element of the symmetry group G. So we find that the phase space of all possible distinct manifolds of this type is equivalent[1] to $\mathrm{Hom}(\pi_1(\Sigma), G)/G$: the set of all maps $\pi_1(\Sigma) \to G$ modulo an overall conjugation by an arbitrary element of G. For Σ a genus g handle body we are embedding $2g$ generators, with two degrees of freedom removed by the group relation and by modding out conjugations, into a six-dimensional symmetry group. This defines a space of classical solutions with dimension $12g - 12$.

We must note before moving on that in fact the above construction produces an over-counting of classical solutions of the theory. This is because quotienting by G only ensures that our solutions are distinct up to diffeomorphisms that are continuously connected to the identity—that is, those that can be generated by elements of G. For topologically nontrivial manifolds there will also be large diffeomorphisms, not deformable to the identity, that can connect different elements of the above solution space. Since general relativity is meant to be invariant under all diffeomorphisms, we must take these into account by quotienting our solution space by a discrete group Γ: the group of large diffeomorphisms (modulo infinitesimally generated ones) of M. Γ is known as the mapping class group of M. We will study such groups in more detail in Chapter 5.

[1] There is actually a slight subtlety here. The space $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ is in fact several disjoint spaces labelled by the Euler character of the map, and only those maps with maximal Euler character produce well-behaved topological spaces. The actual space of solutions is this disjoint subset of the full space of maps [8]. Furthermore, this space must then be quotiented by the mapping class group.

Taking all this into consideration, we have found a definition of our phase space of classical solutions for a $\mathbb{R} \times \Sigma$ universe of genus g. It is a rather formal definition, however, and presents some problems in terms of quantization. For example, it is not clear from this definition of the phase space which degrees of freedom ought to be thought of as 'position' variables and which to associate with their canonical momenta; we therefore have no Poisson bracket with which to define canonical commutation relations in the quantum theory. For this there are two formalisms that we can turn to. The first is the ADM formalism [10] as applied to compact 3d universes by Moncrief [11], which we will find useful for gaining an intuitive picture of the phase spaces we are considering. The second is the Chern-Simons formulation of 3d gravity, originally discovered by Achúcarro and Townsend [2] and later rediscovered and expanded by Witten [3], which will be useful because it extends very easily to a parity-violating theory.

2.2 The ADM formalism

The ADM formalism, named for its originators Arnowitt, Deser and Misner [10], proceeds by choosing a time-coordinate along which to slice the spacetime, and then decomposing the metric as

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \tag{2.5}$$

Here N and N^i are known as the lapse and shift functions, respectively, and g_{ij} is the induced metric on a spatial hyper-surface, which in our case will just be the Riemann surface Σ.

Let us note here for the first time a fact about two-dimensional geometry which will be of much use to us: any metric on a Riemann surface is conformally related to a metric of constant curvature. This is known as the uniformization theorem. In particular, for Σ a compact surface of genus $g > 1$ with a given metric g_{ij} there exists [11] a function $\lambda(x^i)$ on Σ such that

$$g_{ij} = e^{2\lambda}\bar{g}_{ij}(m_\alpha) \tag{2.6}$$

where $\bar{g}_{ij}(m_\alpha)$ is a metric on Σ of constant curvature $k = -1$. m_α are the moduli of $\bar{g}_{ij}$: a set of $6g - 6$ parameters that completely determine a $k = -1$ constant-curvature metric on a genus g Riemann surface. The space of these moduli—which is therefore also the space of such metrics—is called moduli space, denoted $\mathcal{M}_g$.

For the metric decomposition (2.5), the Einstein-Hilbert action takes the form

$$I = \frac{1}{16\pi G}\int dt \int_\Sigma d^2x(\pi^{ij}\dot{g}_{ij} - N^i\mathcal{H}_i - N\mathcal{H}). \tag{2.7}$$

Here we find the conjugate momentum to g^{ij}: $\pi^{ij} = \sqrt{g}(K^{ij} - g^{ij}K)$, where K^{ij} is the extrinsic curvature of the spatial surface and $K = K_i{}^i$. Meanwhile, the other two terms in the action are defined by [11]

$$\mathcal{H}_i = -2\nabla_j \pi^j{}_i \tag{2.8}$$

$$\mathcal{H} = \frac{1}{\sqrt{g}}g_{ij}g_{kl}(\pi^{ik}\pi^{jl} - \pi^{ij}\pi^{kl}) - \sqrt{g}({}^{(2)}R - 2\Lambda) \tag{2.9}$$

where ${}^{(2)}R$ is the Ricci scalar on Σ. The terms in the action containing $\mathcal{H}_i$ and $\mathcal{H}$ are nondynamical, so varying with respect to the lapse N and shift N_i simply yields the constraint that both (2.8) and (2.9) must vanish for any classical solution. Moncrief [11] has shown that the first of these, known as the momentum constraint, is trivially solved if we choose the York time-slicing [12] which takes Σ to be a surface of constant curvature K and chooses the time-coordinate $T = -K = g_{ij}\pi^{ij}/\sqrt{g}$. The second can be solved by decomposing g_{ij} according to (2.6) and choosing λ to satisfy [11]

$$2\bar{\Delta}\lambda - \frac{1}{2}(T^2 - 4\Lambda)e^{2\lambda} + \left[\bar{g}^{-1}\bar{g}_{ij}\bar{g}_{kl}p^{ik}p^{jl}\right] - k = 0 \qquad (2.10)$$

where as previously stated, for our purposes $k = -1$.

The tensor $p^{ij} = e^{2\lambda}(\pi^{ij} - \frac{1}{2}T\sqrt{g}g^{ij})$ is the transverse-traceless part of the momentum conjugate to $\bar{g}_{ij}$, the trace of the momentum having been determined by the York time-slicing. Much like $\bar{g}_{ij}$ it is completely determined by a set of parameters p^α defined by

$$p^\alpha = \int_\Sigma d^2x\, p^{ij}\frac{\partial}{\partial m_\alpha}\bar{g}_{ij}. \qquad (2.11)$$

These p^α are precisely the conjugate momenta to m_α [11], so they satisfy the classical Poisson brackets

$$\{p^\alpha, m_\beta\} = \delta^\alpha_\beta, \;\; \{m_\alpha, m_\beta\} = \{p^\alpha, p^\beta\} = 0. \qquad (2.12)$$

A choice of values (m_α, p^α), $\alpha = 1, \ldots, 6g - 6$ uniquely determines λ, $\bar{g}_{ij}$ and π^{ij}, and so specifies a unique classical solution to the equations of motion. We see therefore that these are the coordinates of the classical phase space, which in agreement with our result in the previous section is of dimension $12g - 12$ and which we now know to be isomorphic to $T^*\mathcal{M}_g$, the cotangent bundle of moduli space.

Ideally from this point one would simply perform the standard quantization procedure of turning these canonical variables into operators and converting the Poisson bracket into the commutator, $[m_\alpha, p^\beta] = i\hbar\delta^\beta_\alpha$. There are some problems with this, however. Chief among them is the fact that the Hamiltonian constraint has left us with the following highly nontrivial, time-dependent Hamiltonian on phase space [11]:

$$H = \int_\Sigma d^2x\sqrt{g}e^{2\lambda(m,p,T)} \tag{2.13}$$

In the case where Σ is a torus, this Hamiltonian can be written out explicitly, and the resulting eigenfunctions on phase space have been discussed by Puzio in [13]. For higher genus, however, the Hamiltonian becomes more complicated and this approach is likely to be intractable [14]. However, there is an alternate formulation of the problem which, in addition to allowing us to see the problem in yet another new light, also avoids the problem of a complicated Hamiltonian by parametrizing the phase space in terms of topological invariants that do not evolve with time. We therefore turn to an exposition of the Chern-Simons formulation of 3d gravity.

2.3 3d gravity as a Chern-Simons theory

The basis of the Chern-Simons formulation of 3d gravity is the spin-connection formalism of general relativity, which uses the triad ${e_\mu}^a$ and spin connection ${\omega_\mu}^{ab}$ as its fundamental degrees of freedom instead of working directly with components of the metric. The triad ${e_\mu}^a$ can be thought of as a change-of-basis transformation on the tangent space at any point on M, which takes us from the basis defined by the coordinates to an arbitrary basis of our choice. It is more formally defined [3] as an isomorphism from the tangent bundle of the manifold to an abstract vector bundle V. V has an $SO(2,1)$ group structure and is endowed with a Minkowski metric η_{ab}, which we can use to raise and lower roman indices on the triad and other V-valued objects, as well as a volume 3-form ϵ_{abc}. The metric on the manifold is determined by the equation

$g_{\mu\nu} = \eta_{ab} e_\mu{}^a e_\nu{}^b$; this often leads to the characterization of the triad as the 'square root' of the metric. We can also think of the triad as a V-valued one-form $e^a = e_\mu{}^a dx^\mu$. The spin connection $\omega_\mu{}^{ab}$ is so named because it appears as the connection on covariant derivatives of spinor fields on curved manifolds. We can think of it as the connection that defines parallel transport on the vector bundle V. The spin connection can be thought of as an antisymmetric $V \otimes V$-valued one-form $\omega^{ab} = \omega_\mu{}^{ab} dx^\mu$. By using the volume tensor we can also represent it also as a V-valued one-form $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{\mu bc} dx^\mu$.

In the spin-connection formalism, the action is given by

$$I = -\frac{1}{8\pi G}\int_M \left\{ e^a \wedge \left(d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c \right) + \frac{\Lambda}{6}\epsilon_{abc} e^a \wedge e^b \wedge e^c \right\}. \qquad (2.14)$$

We regard ω and e as completely independent variables and vary I with each of them in turn. This yields the equations of motion

$$de_a + \epsilon_{abc}\omega^b \wedge e^c = 0, \qquad (2.15)$$

$$d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c = -\frac{\Lambda}{2}\epsilon_{abc} e^b \wedge e^c. \qquad (2.16)$$

The first equation determines ω as a torsion-free connection and completely fixes it as a function of the triad,

$$\omega_\mu{}^a = \epsilon^{abc} e^\nu{}_c (\partial_\mu e_{\nu b} - \partial_\nu e_{\mu b}) - \frac{1}{2}\epsilon^{bcd}(e^\nu{}_b e^\rho{}_c \partial_\rho e_{\nu d}) e_\mu{}^a \qquad (2.17)$$

which is in fact how the spin connection was historically defined. With this definition, the Riemann tensor can be written as a V-valued 2-form which is in fact precisely the left side of (2.16), and this equation turns out to be precisely equivalent to the vacuum Einstein equations with cosmological constant (2.2). So the spin connection formalism and classical Einstein gravity are in fact completely equivalent theories. For more detail on this, see [8],[3].

We now proceed by treating both ω and e as gauge fields. Witten [3] has shown that for any value of Λ, we can construct gauge fields out of ω and e such that the action (2.14) becomes precisely the Chern-Simons action

$$I_{CS} = \frac{k}{4\pi} \int_M \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{2.18}$$

for a gauge field A with gauge group equal to the local diffeomorphism group of the theory (the symmetry group of the universal covering space of classical solutions that we discussed in Section 2.1). This makes sense intuitively, since like 3d gravity Chern-Simons theory is a topological theory; it has no local degrees of freedom. It also represents a vast improvement from the ADM formalism, because the Hamiltonian of a Chern-Simons theory vanishes identically [15]. So there will no longer be any need to diagonalize a complicated time-dependent Hamiltonian to understand the quantum theory. Rather, the theory is fully characterized by the Poisson bracket imposed on the phase space by the action (2.18). This simplicity, which springs from the fact that the observables in Chern-Simons theory are all topological invariants, comes at the cost of making it very difficult, and perhaps impossible [14] to reconstruct the evolution of the spatial metric that was so explicitly seen in the ADM formalism. Luckily, this will not be necessary for our purposes.

We calculate the Poisson bracket for a Chern-Simons theory in the usual way, by choosing a time-direction and calculating the conjugate momenta of field components by differentiating the Lagrangian by their time-derivatives. The result is [16]

$$\{A_i{}^a(x), A_j{}^b(x)\} = \frac{4\pi}{k} \mathrm{tr}(T_a T_b) \epsilon_{ij} \delta^{(2)}(x-y). \tag{2.19}$$

Here T_a are the generators of the gauge group representation and $\mathrm{tr}(T_a T_b)$ is the bilinear form that defines the trace in (2.18). To quantize the theory, we simply perform the usual operation of converting the components of A into operators and converting the Poisson bracket into the commutator

$$[\hat{A}_i^a(x), \hat{A}_j^b(x)] = -\frac{4\pi i\hbar}{k}\mathrm{tr}(T_a T_b)\epsilon_{ij}\delta^{(2)}(x-y). \tag{2.20}$$

The construction of the gauge field is somewhat different depending on the value of the cosmological constant, so we will focus only on the case where Λ is negative. In this case the gauge group will be $SO(2,2)$, which is the same as $SO(2,1) \times SO(2,1)$. So we can decompose the gauge field action (2.18) into the actions of two $SO(2,1)$ gauge fields, left and right, which we will call A_L and A_R:

$$\begin{aligned} &I_{CS} = k(I_L - I_R) \\ &= \frac{k}{4\pi}\int_M \mathrm{tr}(A_L \wedge dA_L + \frac{2}{3}A_L \wedge A_L \wedge A_L) - \frac{k}{4\pi}\int_M \mathrm{tr}(A_R \wedge dA_R + \frac{2}{3}A_R \wedge A_R \wedge A_R) \end{aligned} \tag{2.21}$$

The reason for the relative minus sign will soon become apparent. Let us now define the generators T_a of a three-dimensional representation of $SO(2,1)$, satisfying the commutation relations

$$[T_a, T_b] = \epsilon_{abc}T^c. \tag{2.22}$$

where again we are raising and lowering roman indices with η_{ab}.The trace on these generators is uniquely determined by the bilinear form

$$\mathrm{tr}(T_a T_b) = \eta_{ab}. \tag{2.23}$$

We now define our $SO(2,1)$ gauge fields as

$$A_L = (\omega^a - e^a/l)T_a, \tag{2.24}$$

$$A_R = (\omega^a + e^a/l)T_a. \tag{2.25}$$

where l is a positive length scale defined by $\Lambda = -1/l^2$. With these definitions and using (2.22) and (2.23) to evaluate the trace, the expressions for the left and right Chern-Simons actions become

$$\begin{aligned} I_{R/L} = \frac{1}{4\pi}\int_M \left\{\omega^a \wedge d\omega_a + \frac{1}{3}\epsilon_{abc}\omega^a \wedge \omega^b \wedge \omega^c - \frac{1}{l^2}\left(e^a \wedge de_a + \epsilon_{abc}\omega^a \wedge e^b \wedge e^c\right)\right\} \\ \pm \frac{1}{4\pi}\int_M \frac{1}{l}\left\{2e^a \wedge d\omega_a + \epsilon_{abc}e^a \wedge \omega^b \wedge \omega^c - \frac{1}{3l^2}\epsilon_{abc}e^a \wedge e^b \wedge e^c\right\}. \end{aligned} \tag{2.26}$$

In this form, it is easy to see that with the identification $k = l/16G$, our Chern-Simons action (2.21) is precisely the first-order gravitational action (2.14).

Now just as we did in the case of the ADM formalism, we would like to apply the classical constraints to our field equations and and find the resulting classical phase space of solutions. For any Chern-Simons theory with action (2.18), the classical constraint equations take the simple form [16]

$$F_\Sigma = (dA + A_\wedge A)_\Sigma = 0 \tag{2.27}$$

where the Σ subscript denotes that this is the curvature of the gauge connection confined to the spatial slice Σ. In our case the decomposition of the gauge group simply requires us to enforce $F_{L\Sigma} = F_{R\Sigma} = 0$. We see now another great advantage of working in the Chern-Simons formalism: the constraints (2.8) and (2.9) that appeared so complicated in the ADM formalism are here reduced to the very simple requirement that the gauge field must have vanishing curvature.

Any gauge connection A with this property will specify a classical solution of the theory. In addition, as is usual in gauge theory two connections are considered to specify the same solution if they differ by a gauge transformation. Thus the phase space $\mathcal{M}$ of our Chern-Simons theory is the set of all such $SO(2,2)$ gauge connections on M, quotiented by the action of the gauge group [3]. The Poisson bracket (2.19) descends trivially from the infinite-dimensional space of all possible connections onto a Poisson bracket on this phase space [17]. Flat gauge connections are completely determined [15] by their holonomies around nontrivial cycles of M: for $M \approx \mathbb{R} \times \Sigma$, they are in fact in one-to-one correspondence with the embeddings of $\pi_1(\Sigma)$ into the gauge group. Quotienting by gauge transformations thus brings us back to $\mathcal{M} = \mathrm{Hom}(\pi_1(\Sigma), SO(2,2))/SO(2,2)$. This agrees exactly with the result we obtained in section 2.1, except for the conspicuous absence of the additional quotient by Γ, the mapping class group of M. This shows us that 3d gravity is in fact only locally equivalent to a Chern-Simons theory, with additional global considerations that must be put in by hand. We will shortly see how to reintroduce the mapping class group.

Ignoring this complication for a moment, we turn to examine our Chern-Simons phase space $\mathcal{M}$ which in this case factors into $\mathcal{M}_L \times \mathcal{M}_R$, where $\mathcal{M}_L = \mathcal{M}_R = \mathrm{Hom}(\pi_1(\Sigma), SO(2,1))/SO(2,1)$ is the space of flat $SO(2,1)$ connections on Σ. $SO(2,1) = PSL(2,R)$ is the symmetry group of the 2-dimensional hyperbolic plane $\mathbb{H}^2$, the universal covering space of Riemann surfaces with constant negative curvature. So as we saw in section 2.1, specifying a given embedding of $\pi_1(\Sigma)$ into $SO(2,1)$ is equivalent to specifying a constant negative curvature metric on Σ, and the space embeddings is isomorphic to the space of such metrics. As before, these metrics are distinct only up to infinitesimally generated diffeomorphisms. This space of constant negative-curvature metrics on a genus g Riemann surface has been well studied and is known as Teichmüller space, denoted $\mathcal{T}_g$. We therefore obtain the result that, locally, our phase space is $\mathcal{M} = \mathcal{T}_g \times \mathcal{T}_g$.

In section 2.2 we obtained the expression $T^*\mathcal{M}_g$ for our phase space, where $\mathcal{M}_g$ is the moduli space of constant negative curvature metrics on Σ modulo *all* diffeomorphisms. It is easy to see that this implies that $\mathcal{M}_g = \mathcal{T}_g/\Gamma$. This gives us a clue as to how to act with the mapping class group on $\mathcal{M}$: we simply have to find the slice through $\mathcal{M}$ that describes the same solutions as the trivial section of $T^*\mathcal{M}_g$—i.e. those where all the momenta p^α have been set to zero. These solutions are the *static moduli* solutions: time-reversal-symmetric metrics which can be written as

$$ds^2 = 4l^2 \left(-dt^2 + \frac{1}{2}\cos^2(t)\bar{g}_{ij}dx^i dx^j \right) \tag{2.28}$$

where $\bar{g}_{ij}$ is an arbitrary metric on Σ with curvature $k = -1$. They form a $6g - 6$ dimensional solution space that is obviously isomorphic to $\mathcal{M}_g$. In the Chern-Simons formulation, we restrict to the equivalent subset of solutions by imposing the constraint

$$A_{L\mu}{}^a = (-1)^{\mu+a} A_{R\mu}{}^a. \tag{2.29}$$

Any solution of the form (2.28), with spin connection given by (2.17), will satisfy this constraint. So the space of static moduli solutions is here represented by the degrees of freedom of a single $SO(2,1)$ gauge field, and therefore by yet another Teichmüller space $\mathcal{T}_g$: a diagonal slice through $\mathcal{M}$ constructed out of components of both $\mathcal{M}_L$ and $\mathcal{M}_R$. To connect this result with the space we got from the ADM formalism, we see that our quotient by the mapping class group must act internally on this slice, only identifying its points with other points on the same slice. This slice then becomes the base $\mathcal{M}_g = \mathcal{T}_g/\Gamma$ of the cotangent bundle $T^*\mathcal{M}_g$, and the remaining degrees of freedom make up the fibre.

2.4 Parity-violating Chern-Simons gravity

We would now like to look at a small extension to Einstein gravity that arises naturally from the Chern-Simons formulation. It is obtained by allowing different coefficients for the components I_L and I_R in the gravitational action [18], i.e. a generalized Lagrangian of the form

$$I = k_L I_L - k_R I_R = \frac{k_L + k_R}{2}(I_L - I_R) + (k_L - k_R)\frac{I_L + I_R}{2}. \tag{2.30}$$

The first term in the final expression is the usual Einstein-Hilbert action with $k = \frac{k_L+k_R}{2}$. The second is the so-called 'exotic' term, and is parity-violating. Since this term is also the sum of two $SO(2,1)$ Chern-Simons actions, it must also be possible to write it as a second $SO(2,2)$ Chern-Simons action of the form (2.18). Witten [18] has shown that this is indeed the case. Written out explicitly in terms of e and ω, the exotic term is given by

$$\Delta I = \frac{k'}{4\pi}\int_M \left\{\omega^a \wedge d\omega_a + \frac{1}{3}\epsilon_{abc}\omega^a \wedge \omega^b \wedge \omega^c - \frac{1}{l^2}\left(e^a \wedge de_a + \epsilon_{abc}\omega^a \wedge e^b \wedge e^c\right)\right\} \tag{2.31}$$

where we have defined $k' = k_L - k_R$. Somewhat incredibly, varying this new action with respect to e and ω produces the exact same system of equations (2.15) and (2.16) as before, so the extended theory has all the same solutions and the exact same classical phase space as the old one [3].

Despite their identical phase space of solutions, the addition of the exotic term to the gravitational action will have profound implications for the quantization of the theory. An easy illustration of this, and one which we will be spending a large portion of this work discussing, is to see what happens when we again restrict ourselves to the static moduli solutions by imposing the condition (2.29), and look at the resulting theory on the reduced phase space $\mathcal{M}_g$. It is easy to verify that the $SO(2,1)$ Chern-Simons action has a discrete symmetry under ${A_\mu}^a \to (-1)^{\mu+a}{A_\mu}^a$. So the condition (2.29) implies that, on this reduced slice of phase space, $I_L = I_R$. In the case where there is no exotic term in the action these terms will exactly cancel, leaving us with an action that is identically zero! The theory is trivial on this reduced slice of phase space; no commutators can be defined, and there is no quantum mechanics to be done. This is in line with the ADM picture: in our attempt to quantize ADM phase space the only nonzero commutators are between moduli and their conjugate momenta, so reducing to the subspace where all momenta vanish leaves us with no nonzero commutation relations, and a trivial quantum mechanics.

The situation is much different if $k' = k_L - k_R$ is nonzero, because in the exotic term I_L and I_R are added rather than subtracted. The result is that restricting to the subspace of static moduli solutions no longer kills the gravitational action, but transforms it into the action of a single $SO(2,1)$ Chern-Simons field—the same field, in fact, that we used to construct our diagonal Teichmüller space in the first place. So our reduced phase space is endowed with the usual Poisson bracket (2.19). The resulting theory, however, is not simply a quantized Chern-Simons theory, because our phase space is not the usual phase space $\mathcal{T}_g$; it has been modded out by the action of the mapping class group to yield $\mathcal{M}_g$.

Chapter 3

Statement of the Problem

3.1 A minisuperspace model

We now come to the problem that we wish to consider: the quantization of $\Lambda < 0$ parity-violating 3d gravity on a compact $\mathbb{R} \times \Sigma$ universe, restricted to the reduced phase space of static moduli solutions. These solutions may be considered to be in some sense the 'ground states' of the theory: they are the solutions of greatest symmetry, being the only solutions to be time-reversal invariant, and in the ADM formalism they are the solutions with all momenta set to zero. Our restriction to this subset of the phase space may be thought of as a much less drastic version of the minisuperspace model considered by Hartle and Hawking in their study of the wave function of the universe [19]. There the authors reduce the infinite-dimensional phase space of (3+1)-dimensional gravity with a free scalar field to its subspace of maximally symmetric solutions: a two-parameter family of homogeneous, isotropic closed universes characterized solely by their time-dependent scale-factor $a(t)$ and isotropic field-value $\phi(t)$. They restrict the Einstein-Hilbert action to vary only in these two parameters and quantize the resulting theory. Our construction, by comparison is much less extreme: We are only reducing the phase space by half its degrees of freedom, from the $12g - 12$ free parameters of the full solution space to the $6g-6$ dimensions of $\mathcal{M}_g$. Nevertheless, in both cases the motivation is the same: we are reducing a difficult quantum mechanics problem to a simpler one that still contains enough interesting physics to provide some insight into solutions of the full theory. In Hartle and Hawking's case, the result is a ground-state wave function in terms of their two variables [19] which they relate to the ground state of the full quantum theory. In our case, our reduced model provides a test of whether the full theory can be quantized at all.

The potential impediment to the quantization of our parity-violating gravity theory can be traced back to the large-diffeomorphism invariance of gravity. It is this invariance that forces us to quotient our Chern-Simons solution space by the mapping class group Γ. This transforms the phase space from a well-behaved manifold into a topologically nontrivial space with orbifold singularities. In general such phase spaces are not guaranteed to be quantizable because the wave functions obtained in solving the quantum theory are multivalued; they do not return to themselves around nontrivial cycles. However, when a coupling constant is present in the commutator, as it is in our case, we can always tune the coupling to specific quantized values that will make the wave functions single-valued. This provides a quantization condition for the coupling constant. Thus, the first interesting question we could ask is what the quantization condition is for our exotic-term coupling constant k', and how it changes with the topology of the space we consider. Since the mapping class group acts internally on the subspace of static moduli solutions, this subspace will contain all possible orbifold points and our reduced phase space scheme should be sufficient to determine this quantization condition. We will prove that the quantization condition we obtain is at least a necessary constraint on the full theory, if not a sufficient one.

The quantization of k' is not in itself a problem. In fact, the coupling constants of Chern-Simons gauge theories are usually quantized as an artifact of how the action of the theory is rigorously defined. See, e.g. Killingback [15] for an explanation of this. This phenomenon only applies to gauge groups with nontrivial third homotopy group π_3, and since $\pi_3(SO(2,1))$ is trivial [15] our coupling constant is not quantized by this mechanism. However, Witten's work on the 3d AdS/CFT correspondence [18] suggests that such couplings may have to be quantized anyway when working in the context of Chern-Simons gravity, both to make contact with the quantized central charges of boundary CFTs and to enable analytic continuation to Euclidian gravity, for which the local symmetry group transforms into $SO(3)$. In any case, it would certainly seem that a quantization condition is not particularly problematic for Chern-Simons theories.

The new factor introduced here is that a theory of gravity should not be confined to a single space-time topology, but should be sensibly defined for arbitrary topologies. In the context of our study of compact universes, this means that we should be able to apply the same theory—with the same coupling constants—to universes with compact spatial slices of arbitrarily high genus. So we have not a single quantization condition but an infinite number of them, one for each value of g. The question then becomes, are all of these quantization conditions mutually compatible? One could easily imagine a scenario where, as g becomes large and $\mathcal{M}_g$ acquires higher and higher-degree orbifold points, the smallest permissible nonzero value for k' might grow without limit. In this case the theory becomes nonsensical and the only possibility is that $k' = 0$, reducing us to vanilla (2+1)-dimensional Einstein gravity where the commutator on $\mathcal{M}_g$ is trivial. If, on the other hand, the quantization conditions turn out to be compatible, and the full theory quantizable, we would like to understand to the best of our ability the resulting quantum Hilbert space and its dependence on the topology of the space.

To answer these questions, we first have to understand geometric quantization: the quantization scheme that will allow us make rigorous statements about the quantizability and Hilbert space of a quantum theory for which all we have is a phase space and a commutator. We will then need to gain an understanding the geometry on $\mathcal{M}_g$ and the nontrivial cycles that will determine whether or not we can construct single-valued wave functions to live on it. Expositions on these topics will therefore be the topics of our next two chapters.

3.2 Particle in a uniform magnetic field

Before proceeding, we present a small exposition on a completely different problem: the quantization of a quantum particle moving in a uniform magnetic field. This will provide a fully solvable 'baby example' of the effects of adding a parity-violating term to the action of quantum theory, and its effects on the quantization of the subspace of ground states of the theory. We will also find this problem illuminating as an illustrative example during our discussion of geometric quantization. Results in the following are taken from [20], [21].

3.2.1 The free particle

Consider a free charged particle confined to a two-dimensional plane with coordinates (x, y). In the absence of fields, the classical action of this particle is

$$S = \frac{1}{2} \int dt(\dot{x}^2 + \dot{y}^2). \tag{3.1}$$

It is easy to see that $p_x = \Pi_x = \dot{x}$ and $p_y = \Pi_y = \dot{y}$ are the canonical momenta conjugate to x and y respectively, where throughout this discussion we will use $\mathbf{\Pi}$ to denote the kinetic momentum $\dot{\mathbf{x}}$, reserving the symbol $\mathbf{p}$ for the canonical momentum. The Hamiltonian is, of course

$$H = \frac{1}{2}\mathbf{\Pi}^2 = \frac{1}{2}\mathbf{p}^2. \tag{3.2}$$

The phase space of classical solutions is the tangent bundle $T^*\mathbb{C}$ with coordinates (x, y, p_x, p_y). We quantize the phase space by turning its coordinates into operators and imposing the canonical commutation relations

$$[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar \tag{3.3}$$

These commutators are analogous to the ones we found for parity-preserving Einstein gravity in section 2.2. Note that if we reduce our phase space to its base by imposing the condition $p = 0$, all of the commutators become trivial.

As any first-year student of quantum mechanics knows, the wave-functions that diagonalize the Hamiltonian are kinetic momentum eigenfunctions $|\Pi_x, \Pi_y\rangle$, completely characterized by their momenta. We mention this only to note that there is a unique ground state: the Hamiltonian eigenstate corresponding to $\langle\hat{\Pi}_x\rangle = \langle\hat{\Pi}_y\rangle = 0$ for which the position wave function is a constant. This uniqueness will not be the case when we add a magnetic field.

3.2.2 Addition of a uniform magnetic field

Now consider the addition of a uniform magnetic field of strength B, pointing in the z direction. It can be thought of as arising from the gauge potential

$$\mathbf{A} = (A_x, A_y) = \frac{B}{2}(-y, x). \tag{3.4}$$

The new action is given by

$$S = \int dt \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \mathbf{A} \cdot \dot{\mathbf{x}} \right). \tag{3.5}$$

The extra term in the action is a parity-violating term. Taking the variation with respect to the coordinates, the conjugate momenta of x and y are no longer equal to the kinetic momenta $\dot{x}$ and $\dot{y}$. Instead they are given by

$$(p_x, p_y) = (\Pi_x - \frac{B}{2}y, \Pi_y + \frac{B}{2}x) \tag{3.6}$$

while the Hamiltonian remains

$$H = \frac{1}{2}\mathbf{\Pi}^2 = \frac{1}{2}\left(\mathbf{p} - \mathbf{A}\right)^2 = \frac{1}{2}\left(\mathbf{p}^2 + \frac{B^2}{4}\mathbf{x}^2 + \frac{B}{2}(p_x y - p_y x)\right). \tag{3.7}$$

To quantize this system, we again turn our observables into operators and impose the canonical commutation relations (3.3). Note that the form of the commutators has not changed, but the definition of $\hat{p}$ has. Now the Hamiltonian cannot be diagonalized by eigenfunctions of both $\hat{\Pi}_x$ and $\hat{\Pi}_y$, since these operators do not commute. Rather, they satisfy

$$[\hat{\Pi}_x, \hat{\Pi}_y] = i\hbar B. \tag{3.8}$$

Because of this we can construct creation and annihilation operators which satisfy the commutation relation $[a, a^\dagger] = 1$:

$$\hat{a} = \sqrt{\frac{1}{2\hbar B}}(\hat{\Pi}_x + i\hat{\Pi}_y) \tag{3.9}$$

$$\hat{a}^\dagger = \sqrt{\frac{1}{2\hbar B}}(\hat{\Pi}_x - i\hat{\Pi}_y) \tag{3.10}$$

The Hamiltonian then becomes

$$\hat{H} = \hbar B \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right). \tag{3.11}$$

We can thus construct all the eigenfunctions of the Hamiltonian by first finding all wave functions annihilated by $\hat{a}$ and then acting on them with $\hat{a}^\dagger$. To do this we define complex coordinates $z = x + iy$ and $\bar{z} = x - iy$. In these coordinates the annihilation condition takes the form

$$\langle z, \bar{z} | \, \hat{a} \, |0\rangle = -i\sqrt{\frac{\hbar}{2B}} \left(2\bar{\partial} + \frac{B}{2\hbar} z \right) \psi(z, \bar{z}) = 0 \tag{3.12}$$

and the solution wave functions are

$$\psi(z, \bar{z}) = f(z) e^{-B\bar{z}z/4\hbar} \tag{3.13}$$

where $f(z)$ is any holomorphic function of z, so the space of ground state wave functions is isomorphic to the space of holomorphic functions on the complex plane. Thus we have the surprising result that, unlike the free particle which had a single state representing a stationary particle, here we have an infinite tower of lowest-energy states, each of which is as good a candidate as any other for the 'best approximation' of a particle at rest.

We can generate our infinite tower of ground states by considering the restriction of the theory onto the slice of the phase space with $\Pi_x{=}\Pi_y = 0$, i.e. the complex plane $\mathbb{C}$, and imposing a priori the commutation relation

$$[\hat{x}, \hat{y}] = -\frac{i\hbar}{B}. \tag{3.14}$$

How do we define these operators in terms of their action on wave functions? One way would be to arbitrarily decide to treat one as the canonical momentum of the other, defining $\hat{y} \equiv \frac{i\hbar}{B}\partial_x$ or $\hat{x} \equiv -\frac{i\hbar}{B}\partial_y$. More equitably, we can treat our two coordinates on the same level and define instead

$$\hat{x} = \frac{1}{2}x - \frac{i\hbar}{B}\partial_y, \quad \hat{y} = \frac{1}{2}y + \frac{i\hbar}{B}\partial_x. \tag{3.15}$$

It is easy to see that these definitions satisfy the commutator (3.14).

From these operators we can construct another set of creation-annihilation operators $\hat{b}$ and $\hat{b}^\dagger$, which satisfy $[\hat{b}, \hat{b}^\dagger] = 1$:

$$\hat{b} = \sqrt{\frac{B}{2\hbar}}(\hat{x} - i\hat{y}) = \sqrt{\frac{\hbar}{2B}}\left(2\partial + \frac{B}{2\hbar}\bar{z}\right) \tag{3.16}$$

$$\hat{b}^\dagger = \sqrt{\frac{B}{2\hbar}}(\hat{x} + i\hat{y}) = \sqrt{\frac{\hbar}{2B}}\left(-2\bar{\partial} + \frac{B}{2\hbar}z\right) \tag{3.17}$$

The only Hamiltonian ground-state that is annihilated by $\hat{b}$ is the one with constant $f(z)$: $\psi_0 = N_0 e^{-B\bar{z}z/4\hbar}$ where N_0 is a normalization constant. By acting with powers of $b^\dagger$, we build up a tower of states

$$\psi_n = (\hat{b}^\dagger)^n \psi_0 = N_n z^n e^{-B\bar{z}z/4\hbar}. \tag{3.18}$$

Since any holomorphic function can be written as a Taylor series, these ψ_n's form a basis for the space of Hamitonian ground states.

Let us recap what we have learned. We have seen that when our commutators paired only base space variables with tangent space variables (the case of the free particle), there was a single ground state corresponding to a constant wave function. However, when we added the parity-violating term, we suddenly gained an infinite degeneracy of ground states corresponding to the holomorphic functions on $\mathbb{C}$. Furthermore, we were able reduce down to the zero-momentum slice of the phase space and still generate the entire tower of ground states by imposing a nontrivial commutator.

Returning to gravity, our proposal is that the same should be true of our parity-violating gravity theory: that the Hilbert space we find by quantizing the base space should be a subspace of degenerate ground states of the full theory. We unfortunately lack the ability so far to explicitly solve the full theory in order to find out if this is correct. However, our problem's similarity to the case of a particle in a magnetic field at least gives us confidence that this proposal makes sense.

Chapter 4

Geometric quantization

We now turn to the problem of quantizing $\mathcal{M}_g$. To make this problem rigorous, we need a systematic quantization scheme that allows us to construct a Hilbert space of wave functions on an arbitrary phase space endowed with a Poisson bracket. This idea finds its realization in the theory of geometric quantization (GQ): the study of the quantization of symplectic manifolds. Here we will provide an overview of the elements of geometric quantization needed for us to analyze the problem at hand. A more complete exposition of the theory of geometric quantization can be found in such reviews as Blau [22] and Hassan and Mainiero [23], as well as references therein. Most of the results in the following chapter are standard results in the field of GQ, and can be found in these reviews unless otherwise noted. The full application of GQ to Chern-Simons gauge theory has been laid out by Axelrod, Della Pietra and Witten in [17].

Geometric quantization begins with an arbitrary manifold endowed with a symplectic structure: a closed non-degenerate two-form that is the GQ equivalent of a Poisson bracket. It begins by choosing a *prequantization*, defined by a certain complex line bundle over the manifold with a specified connection. The existence of such a line bundle is not guaranteed for an arbitrary symplectic manifold; the question of its existence is the GQ equivalent of whether we can define single-valued wave functions on the manifold. If such a line bundle exists the manifold is called *quantizable*, and we can define a *prequantum Hilbert space* as the space of global sections of the line bundle—each section being the GQ equivalent of a wave function. This Hilbert space unfortunately turns out to be much too large to be the proper quantum Hilbert space, and we must reduce its degrees of freedom by picking a *polarization*: a set of constraints on the allowable sections that reduces the Hilbert space to the right number of degrees of freedom.

4.1 Phase space as a symplectic manifold

The starting point of geometric quantization is the symplectic manifold. A symplectic manifold is a manifold M endowed with a two-form ω called a symplectic form. The defining properties of ω are that it must be closed—that is, $d\omega = 0$—and non-degenerate, i.e. it must have non-vanishing determinant at every point on M. Being closed implies that locally ω can always be written as an exterior derivative

$$\omega = d\theta \tag{4.1}$$

of some one-form θ, called the symplectic potential. θ is obviously not unique, since the transformation $\theta \to \theta + d\lambda$ will yield the same symplectic form for any function λ on M. Meanwhile, the non-degeneracy of ω implies that M must be even-dimensional, as the determinant of an odd-dimensional antisymmetric matrix always vanishes. Taking the dimension of the manifold to be $2n$, one can take the nth power of ω to define a volume form, leading to the notion of the symplectic volume of the manifold,

$$\mathrm{Vol}_\omega(M) := \frac{1}{n!}\int_M \omega^n. \tag{4.2}$$

We can think of ω as acting something like a metric, in that it provides an isomorphism between the tangent and cotangent spaces of M. It can be used to associate a one-form, denoted $i(X)\omega$, to every tangent vector X, expressible in local coordinates as

$$i(X)\omega_j = \omega(X, .)_j = \omega_{ij}X^i. \tag{4.3}$$

This then allows us to define a vector field X_f for every smooth function f on M by the relation

$$i(X_f)\omega = -df. \tag{4.4}$$

X_f is called the Hamiltonian vector field of f. Regarding X_f as a differential operator $X_f^i\partial_i$, we see that we have a natural scheme by which a function on a phase space is transformed into a differential operator—exactly what is needed for quantum mechanics. The coordinates of X_f take the explicit form

$${X_f}^i = \omega^{ji}\partial_j f. \tag{4.5}$$

This construction allows us to define the Poisson bracket between two functions f and g according to

$$\{f, g\} = \omega(X_f, X_g) = \omega^{ij}\partial_i f \partial_j g \tag{4.6}$$

The Lie bracket between two Hamiltonian vector fields naturally satisfies the condition

$$[X_f, X_g] = X_{\{f,g\}} \tag{4.7}$$

so, amazingly, our 'operators' turn to have just the right commutation relations, as defined by the Lie bracket, to be the quantum operators corresponding to functions f and g. It will turn out, however, that these are not quite the right operators; they must be supplemented by additional terms to satisfy the properties we would normally like quantum operators to have.

The symplectic manifolds that arise most often in physics are cotangent bundles $T^*\mathcal{Q}$ of some n-dimensional manifold $\mathcal{Q}$. These have a natural set of coordinates: the n dynamical variables q^k that are the coordinates of $\mathcal{Q}$, and their conjugate momenta p_k. A cotangent bundle has a natural symplectic structure given by

$$\omega = dq^k \wedge dp_k. \tag{4.8}$$

On an arbitrary symplectic manifold it is always possible to locally find coordinates in which ω takes this form, but only on cotangent bundles are they guaranteed to be globally defined. If we take (4.8) as our symplectic structure, then the poisson bracket between any functions f and g is given by

$$\{f, g\} = \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q^k} - \frac{\partial f}{\partial q^k}\frac{\partial g}{\partial p_k} \tag{4.9}$$

which is the standard Poisson bracket in Hamiltonian mechanics. Applied to the q's and p's, we get

$$\{p_k, q^l\} = \delta^l_k; \quad \{q^k, q^l\} = \{p_k, p_l\} = 0, \tag{4.10}$$

exactly as expected. On $M = T^*\mathbb{C}$, this gives us the Poisson bracket of a free quantum particle. On $M = T^*\mathcal{M}_g$, it gives us the ADM Poisson bracket of 3d

Einstein gravity.

Different symplectic structures on the same manifold can lead to different quantum theories. For example, to get the quantum theory of a particle in a magnetic field described in section 3.2.2, we again start with $M = T^*\mathbb{C}$ with coordinates (x^i, Π_i), but change the symplectic form to [22]

$$\omega_A = \omega + dA = dx^i \wedge d\Pi_i + Bdx \wedge dy. \tag{4.11}$$

We can easily see that taking $p_i = \Pi_i + A_i$ as defined in equation (3.6), this reduces to the standard form (4.8) of the symplectic structure and gives us the canonical Poisson bracket between x^i and p_i. Furthermore, we see that reducing to the $\Pi = 0$ slice of phase space, we are left with a residual symplectic structure $\omega = Bdx \wedge dy$.

4.2 Prequantization

Having expressed a classical system as a symplectic manifold (M, ω), we wish to find a consistent way of transforming functions f on M into quantum mechanical operators $\hat{f}$ which will act on complex wave functions on M. Our scheme should be such that the operators we define have the following properties:

(a) If a is a constant function, its associated operator is a times the identity.

(b) Linearity: If $f \to \hat{f}$ and $g \to \hat{g}$, then $af + g \to a\hat{f} + \hat{g}$

(c) If $\{f_1, f_2\} = f_3$, then $[\hat{f}_1, \hat{f}_2] = -i\hbar\hat{f}_3$.

Building off of our observations in the previous section, we might first try the assignment $\hat{f} = -i\hbar X_f$. This satisfies (b) and (c), but cannot be the right assignment since it maps any constant function to the null operator. To remedy this, we might try to add the function f itself to our operator; however, the resulting assignment now fails condition (c), despite meeting the others. Instead, the assignment that satisfies all three conditions is

$$\hat{f} = -i\hbar X_f - \theta(X_f) + f. \tag{4.12}$$

where θ is some local symplectic potential.

Let us see how this works in the case of the $\Pi = 0$ slice of $T^*\mathbb{C}$ with symplectic form $\omega = Bdx \wedge dy$. We first calculate the vector fields corresponding to the functions x and y according to (4.5):

$$X_x = \omega^{ji}(\partial_j x)\partial_i = \omega^{xy}\partial_y = \frac{1}{B}\partial_y \tag{4.13}$$

$$X_y = \omega^{ji}(\partial_j y)\partial_i = \omega^{yx}\partial_x = -\frac{1}{B}\partial_x \tag{4.14}$$

Now noting that ω is the total derivative of the symplectic potential $\theta = A = \frac{B}{2}(xdy - ydx)$, we can easily see that $\theta(X_x) = x/2$ and $\theta(X_y) = y/2$. The definition (4.12) then gives us

$$\hat{x} = -i\hbar\left(\frac{1}{B}\partial_y\right) - \frac{x}{2} + x = \frac{x}{2} - \frac{i\hbar}{B}\partial_y \tag{4.15}$$

$$\hat{y} = -i\hbar\left(-\frac{1}{B}\partial_x\right) - \frac{y}{2} + y = \frac{y}{2} + \frac{i\hbar}{B}\partial_x \tag{4.16}$$

in exact agreement with the $\hat{x}$ and $\hat{y}$ operators we defined in section 3.2.2.

We now run into the problem that the above operator assignment is not unique, since as we have seen, θ is defined only up to an arbitrary closed one-form. Any redefinition of θ by the addition of such a form will have no effect on ω, but it clearly *will* change the operators defined by (4.12).

For instance, in the above example we could have just as easily chosen the symplectic potential $\theta = Bxdy$, which would have given us the more 'one-sided' operator assignments $\hat{x} = x$, $\hat{y} = \frac{i\hbar}{B}\partial_x$.

We would like to find a way to define our wave functions such that the entire theory is invariant under such transformations. This can be accomplished if, for the transformation $\theta \to \theta + d\lambda$, the wave function also transforms according to

$$\psi \to e^{i\lambda/\hbar}\psi. \tag{4.17}$$

With this definition, the extra term from the derivative of the wave function will cancel the extra term from the redefinition of θ and the overall action of $\hat{f}$ is invariant. This is a familiar picture: what we have discovered is simply the statement that our theory has a $U(1)$ gauge symmetry. It is therefore most natural to think of ψ not as a complex function, but as a section of a complex line bundle over M, which we will label L. A given choice of λ defines a local trivialization of L, allowing ψ to be written locally as a function. In general though, this trivialization will only be valid for a given domain on M and ψ will undergo transformations of the form (4.17) between trivializations that are valid on different domains. Defining ψ as a section of a pre-existing line bundle ensures that these transformations will all be compatible, so that we can consistently define ψ as a single-valued wave function.

In this picture, changes in the local trivialization also correspond to changes in the symplectic potential θ and the operators $\hat{f}$. Since we have found a manifestly covariant definition for the wave functions, we would like to find an equally covariant definition of these operators. We can accomplish this by regarding θ as a gauge connection that defines to covariant derivative on L:

$$D = d - (i/\hbar)\theta \tag{4.18}$$

With this definition the first two terms of (4.12) are simply a directional derivative along the vector field X_f, and we can write our operator covariantly as

$$\hat{f} = -i\hbar D_{X_f} + f. \tag{4.19}$$

The curvature two-form of D, defined by

$$\Omega(X, Y) = i([D_X, D_Y] - D_{[X,Y]}) \tag{4.20}$$

is readily computed to be

$$\Omega = i(-i/\hbar)d\theta = (1/\hbar)\omega. \tag{4.21}$$

This is in fact the defining property of D, since any covariant derivative on L can be written locally in the form (4.18) if and only if it has curvature $(1/\hbar)\omega$.

We can now rigorously define what we mean by the prequantization of a symplectic manifold (M, ω). A prequantization is defined as a choice (L, D) of complex line bundle L over M and a compatible connection D which is required to have curvature $(1/\hbar)\omega$. A wave function on M is defined as a global section of L, and we define the *prequantum Hilbert space* as the vector space of such sections which are square-integrable with respect to the volume form $\epsilon = \frac{1}{n!}\omega^n$. The operators on this Hilbert space are connection-preserving automorphisms of (L, D), defined by arbitrary functions on M according to the assignment (4.19).

A fact that we will find useful about line bundles is that the tensor product of two complex line bundles is itself a complex line bundle. For two line bundles L and L' with connections $D = d + ia$ and $D' = d + ia'$ in some local trivialization, the connection on the tensor product bundle $L'' = L \otimes L'$ is given by $D'' = d + i(a + a')$: the symplectic potentials simply add. This implies that if (L, D) is a prequantization for (M, ω) and (L', D') is a prequantization for (M, ω'), (L'', D''), as defined above, will be a valid prequantization for $(M, \omega + \omega')$. In particular, this allows us to generate prequantizations for $(M, k\omega)$ for any integer k by taking the the kth power L^k of a valid prequantum line bundle for (M, ω).

The existence of such a pair (L, D) is not guaranteed for an arbitrary symplectic manifold. Consider a local trivialization $\omega = d\theta$, valid over some domain $U \subset M$. It can be shown (see, e.g. Hassan and Mainiero [23]) that the change in the phase of a wave function around some closed curve γ is given by the action of the holonomy operator ξ, defined by

$$\xi = \exp\left(\frac{i}{\hbar}\int_\gamma \theta\right) \tag{4.22}$$

Since $\omega = d\theta$, we can replace $\int_\gamma \theta$ by $\int_\Sigma \omega$ in the above definition, where Σ is any surface such that $\partial\Sigma = \gamma$. There are, however, many choices of possible surface Σ, all of which must lead to the same definition for ξ. So we have the consistency condition that for two choices of surface Σ and Σ', the difference $\int_\Sigma \omega - \int_{\Sigma'} \omega$ must be an integer multiple of $2\pi\hbar$. Alternately, we can define the surface Σ'' as the orientation-reversal of Σ', so that $\partial\Sigma'' = -\gamma$. We can then form a closed surface $S = \Sigma \cup \Sigma''$, and our consistency requirement takes the form

$$\int_S \omega = (2\pi\hbar)n, \quad n \in \mathbb{Z}. \tag{4.23}$$

So we see that to properly define the holonomy operator, which is equivalent to the ability to define a single-valued wave function, the two-form $(1/2\pi\hbar)\omega$ must be in integer cohomology—that is, it must integrate to an integer on any two-cycle in M. This turns out to be both a necessary and sufficient condition for there to exist a prequantization (L, D) for our symplectic manifold. A manifold where this condition holds is said to be *quantizable*, and it is only on such manifolds that geometric quantization can be properly carried out. When (M, ω) is a tangent bundle with the natural symplectic form (4.8), ω can be written globally as a total derivative $d\theta$ with $\theta = p_k dq^k$. Since ω is exact, its integral around any closed two-cycle is identically zero and the above condition is trivially satisfied; these manifolds are therefore always quantizable.

4.3 Polarization

So far our exposition has been fairly straightforward and completely general, applying equally to any quantizable symplectic manifold. This will now be spoiled by the inconvenient fact that the Hilbert space we have defined is not the right one to do quantum mechanics with. For one thing, since we have put no condition on how localized the wave functions are allowed to be, our prequantum Hilbert space is sure to contain states that violate the uncertainty principle. For another, our Hilbert space is simply too large. In the case of a free particle, for example, the allowed wave functions that make up the prequantum Hilbert space are arbitrary square-integrable functions of both the n coordinates q^k and their n conjugate momenta p_k. However, we know that normally in particle quantum mechanics a quantum state is completely specified by its distribution in only n of these parameters: *either* by its spatial wave function *or* its momentum distribution, the two being related by a Fourier transformation. The defining property of these distributions is that they represent the projection of the wave vector $|\psi\rangle$ onto the eigenvectors of a maximal set of commuting observables: $\psi(q) = \langle q|\, \psi\rangle$ and $\psi(p) = \langle p|\, \psi\rangle$ where $|q\rangle$ and $|p\rangle$ are simultaneous eigenstates of all the $\hat{q}^k$ and $\hat{p}_k$ operators, respectively.

We want a construction in the language of geometric quantization that allows us to reduce our Hilbert space to those wave functions that are only functions of a maximal set of commuting observables. The way that GQ accomplishes this is, at first glance, somewhat backward: instead of specifying a set of variables on which our wave functions can depend, we specify a set on which they are *forbidden* to depend. Since locally we can always write ω as $dq^k \wedge dp_k$, we can infer from the free particle case that a maximal set of commuting observables will always be n-dimensional, and therefore so will the set of forbidden variables. The forbidden variables are specified by choosing a *polarization* for our symplectic manifold. Roughly speaking, a polarization is a set of n constraints on the allowed variation of the wave function. We will start by explaining the most intuitive variety of polarization: the real polarization.

4.3.1 Real polarization

To define a real polarization, we choose an n-dimensional sub-bundle of the tangent bundle TM of our symplectic manifold, which we will call P. We then impose the condition that the wave function must have vanishing covariant derivative along any vector field X which is a section of P:

$$D_X \psi = 0 \qquad \forall \; X \in P \tag{4.24}$$

where by $X \in P$ is notational abuse signifying that X is a section of P. This determines at every point a set of n directions along which the wave function must be constant.

The curvature of D imposes certain restrictions on our choice of P. This can be seen by noting that if (4.24) is satisfied, it is certainly true that

$$[D_X, D_Y]\psi = 0 \qquad \forall\, X, Y \in P. \tag{4.25}$$

Combining this with the definition of the curvature form (4.20) and the fact that D has curvature $(1/\hbar)\omega$, we arrive at the condition

$$D_{[X,Y]}\psi - (i/\hbar)\omega(X,Y)\psi = 0 \qquad \forall\, X, Y \in P. \tag{4.26}$$

The easiest way to satisfy this condition is if both terms on the left vanish separately. For the first term, we can accomplish this by demanding that the Lie bracket $[X, Y]$ of two sections of P is itself a section of P. For the second, we simply demand that $\omega(X, Y) = 0$ for all $X, Y \in P$. The first of these conditions is the statement that P is integrable, i.e. that M can be foliated by a set of n-dimensional submanifolds N whose tangent bundles are the restriction of P onto N. The second condition implies that ω vanishes on these submanifolds. A submanifold on which ω vanishes is called *Lagrangian*, and so P is an n-dimensional integrable Lagrangian subbundle of TM. We can take this to be the definition of a real polarization on M.

For any cotangent bundle $M = T^*\mathcal{Q}$ with its natural symplectic form $dq^k \wedge dp_k$, there exists a real polarization called the vertical polarization. Here we take P to be the subbundle of TM whose fibre at any point $p \in M$ is the just the tangent space of the fibre, which is spanned by the vectors $\partial/\partial p_k$. The Lagrangian submanifolds over which ψ must be constant are then just the cotangent spaces $T^*_q\mathcal{Q}, q \in \mathcal{Q}$. So this is a very roundabout way of saying that our wave functions ψ must be functions of the q^k's only, and our Hilbert space reduces to the space of complex-valued functions on $\mathcal{Q}$, exactly as we expect from quantum mechanics.

As intuitive as real polarizations are, they do not exist for all symplectic manifolds (such as the sphere, where real everywhere-nonvanishing vector fields are forbidden by the hairy ball theorem) and in any case are not general enough for our purposes. We can get a more general polarization P by complexifying the tangent bundle $TM \to TM^c$ and defining P to be an n-(complex) dimensional Lagrangian subbundle of TM^c. The quantum Hilbert space is again made up of sections of L that satisfy $D_X\psi = 0$ for all complex vector fields $X \in P$.

4.3.2 Kähler polarization

The properties of completely general polarizations will not concern us, and the reader is encouraged to consult Blau [22] for further details. The only complex polarizations that we will be interested in are Kähler polarizations, which are defined by the property $P \cap \bar{P} = \{0\}$. These are so named because their natural application is to the quantization of Kähler manifolds.

A Kähler manifold is a complex manifold with a symplectic form that is compatible with its complex structure. To understand this definition we must first understand the notion of an almost complex structure. Consider a vector space V. A complex structure on V is a linear map $J : V \to V$ which squares to -1. Being the 'square root' of -1 we can think of J as defining multiplication by i on the vector space, so that general multiplication of a vector $v \in V$ by a complex number is given by

$$(a + ib)v := av + bJv. \tag{4.27}$$

A necessary and sufficient condition for defining J on V is that V must be even-dimensional. Therefore if we take the case where $V = T_pM$ is the tangent space at some point p on M, it is always possible to define a complex structure for T_pM. An *almost complex structure* on M is then defined as a smoothly varying rank (1,1) tensor field J on M with the property that at every point p, J_p is a complex structure on T_pM. If such a J exists, M is called an almost complex manifold. Globally J can be thought of as an isomorphism of the tangent bundle $J : TM \to TM$ that defines the multiplication of vector fields on M by complex numbers. J is called compatible with the symplectic form ω if it satisfies the property

$$\omega(JX, JY) = \omega(X, Y) \tag{4.28}$$

for arbitrary vector fields $X, Y \in TM$.

The complexification V^c of a vector space V (where here we mean complexification in the usual way, independent of J) can always be decomposed into the two eigenspaces $V^{(1,0)}$ and $V^{(0,1)}$ of its complex structure J, with eigenvalues $+i$ and $-i$ respectively. They are spanned by vectors of the form $v \mp iJv, v \in V$. These vector subspaces are complex conjugates of each other, satisfying $V^{(1,0)} \cap V^{(0,1)} = \{0\}$. By decomposing T_pM into such eigenspaces of J_p at every point on M, we can decompose TM^c into two n-dimensional subbundles $T^{(1,0)}M$ and $T^{(0,1)}M$ satisfying the condition

$$T^{(1,0)}M \cap T^{(0,1)}M = \{0\}. \tag{4.29}$$

If these subbundles are integrable then at any point on M we can define holomorphic coordinates on some patch U containing that point, and furthermore the maps between these coordinate on different patches will be holomorphic. A manifold M with these properties is called a complex manifold, and J is then called a complex structure on M. A Kähler manifold is a complex manifold with a symplectic form ω that is compatible with J. In the context of Kähler manifolds ω often referred to as the Kähler form.

If M is a Kähler manifold, then we can easily check that for $X, Y \in TM$,

$$\begin{aligned}
&\omega(X \mp iJX, Y \mp iJY) \\
&\qquad = \{\omega(X,Y) - \omega(JX,JY)\} \mp i\{\omega(X,JY) + \omega(JX,Y)\} \\
&\qquad = 0 \mp i0 = 0. \qquad (4.30)
\end{aligned}$$

Since $T^{(1,0)}M$ and $T^{(0,1)}M$ are spanned by vector fields of this type, this implies that they are Lagrangian subbundles, and so satisfy the criteria to be suitable polarizations on M. The relation (4.29) then shows that both are Kähler polarizations, according to our definition for a Kähler polarization as one for which $P \cap \bar{P} = \{0\}$. In fact, the existence of such a polarization on M is sufficient to prove that that M is a Kähler manifold.

Let us now consider the quantization of a Kähler manifold M. M has holomorphic coordinates z^k with complex conjugates $\bar{z}^k$. In these coordinates, the compatibility of ω with J implies that it takes the form

$$\omega = i\omega_{ij} dz^i \wedge d\bar{z}^j, \quad \bar{\omega}_{ij} = \omega_{ji}. \qquad (4.31)$$

Assuming the existence of a prequantization (L, D), the prequantum Hilbert space is the space of arbitrary sections of L. Under any local trivialization, these sections are arbitrary functions of both the z^k's and the $\bar{z}^k$'s. Taking $T^{(1,0)}M$ as our polarization, which is spanned by the vectors $\partial/\partial\bar{z}^k$, we see that our reduced quantum Hilbert space is simply the set of *holomorphic* sections of L.

It is now practically trivial to see how to get the ground state Hilbert space that we found for the particle in a magnetic field in section 3.2.2. On the reduced phase space $\mathbb{C}$, we simply define our complex structure in the standard way by $J\partial_x = \partial_y, J\partial_y = -\partial_x$. Then taking the usual definitions $z = x+iy, \bar{z} = x-iy$, the tangent subbundles $T^{(1,0)}\mathbb{C}$ and $T^{(0,1)}\mathbb{C}$ are each one-dimensional complex vector bundles spanned by $\bar{\partial} = \partial_x - i\partial_y$ and $\partial = \partial_x + i\partial_y$, respectively. Taking $P = T^{(1,0)}\mathbb{C}$, the polarization condition (4.24) simply becomes the requirement $\bar{\partial}\psi = 0$, and so the reduced quantum Hilbert space is the space of holomorphic functions on $\mathbb{C}$: exactly the Hilbert space we found using standard quantum mechanics.

As a final note, let us consider the large k limiting behaviour of the quantum Hilbert space on a Kähler manifold $(M, k\omega)$ with prequantum line bundle L^k. After Kähler polarization, the quantum Hilbert space is the space of holomorphic sections of L^k, which we denote $H^0(M, L^k)$. If ω is a positive Kähler form, i.e. $\omega(v, Jv) \geq 0 \ \forall \ v \in TM$, then at large k all the higher cohomology groups of L^k vanish by the Kodaira vanishing theorem (see, e.g. [24]), and the Euler character of the line bundle is equal to $\dim H^0(M, L^k)$. It is then a consequence of the Riemann-Roch formula that the dimension of this space takes the limiting value

$$\dim H^0(M, L^k) \to \frac{k^n}{(2\pi\hbar)^n n!}\int_M \omega^n = \left(\frac{k}{2\pi\hbar}\right)^n \mathrm{Vol}_\omega(M) \ \text{ as } \ k \to \infty. \tag{4.32}$$

This is as far into geometric quantization as we will find it necessary to go. It should be noted, however, that we have stopped far short of the full picture. There are numerous issues that we have not addressed. For example, generally the only operators defined by (4.19) that make sense on our reduced Hilbert space will be the ones that map polarized states to polarized states. For vertical polarizations this will imply that we must modify our operator assignment for any function that is not linear in the momenta. This is in fact exactly what we would want, since in normal quantum mechanics these operators are not first-order in derivatives. Another problem is that generally there is no longer any natural notion of square-integrability for our wave functions: for example, the p^k-invariance of vertically polarized wave functions implies that these will all have diverging integral when integrated with respect to the standard measure $\epsilon = \frac{1}{n!}\omega^n$ on M. If we try to remedy this by integrating over the space of leaves of the polarization (i.e. $\mathcal{Q}$ for the vertical polarization), we run into the problem that we have no natural way to determine the measure over this space. This eventually requires us to abandon the picture of wave functions as sections of L and instead view them as 'half forms' on L. We will not be concerned with these details since, in the case of the Kähler polarization, they do not affect the structure of the Hilbert space, which remains the space of Holomorphic sections of L. However, the reader who wishes to understand geometric quantization more fully is referred to references [22] and [23].

4.4 Geometric quantization of 3d gravity

We would now like to ask how all this applies to quantizing the phase space $T^*\mathcal{M}_g$ of our 3d gravity problem. In particular, we would like to see what the condition is for phase space to be quantizable.

For 3d Einstein gravity without a parity-violating term, there is no quantizability condition. As the ADM formalism has taught us, the Poisson bracket in this case is just the standard one for a cotangent bundle, leading to the symplectic form $\omega = dm^\alpha \wedge dp_\alpha$. As usual in such a case, ω is globally the derivative of the symplectic potential $\theta = m^\alpha dp_\alpha$, so it integrates to zero on any two-cycle and trivially satisfies the holonomy condition (4.23). We can polarize our prequantum Hilbert space with a vertical polarization spanned by the vector fields $\partial/\partial p_\alpha$, and the quantum Hilbert space becomes the space of complex functions over $\mathcal{M}_g$, which can also be defined as the space of complex functions on Teichmüller space $\mathcal{T}_g$ which are invariant under the action of the mapping class group Γ. In practice these functions may be hard to find, but in theory at least we have solved the problem of finding a quantum Hilbert space of states in this 3d gravity system.

With the addition of the parity-violating term, things are not so straightforward. Here we must use the first-order formulation of 3d gravity and quantize the resulting Chern-Simons theory. As we have previously seen, the phase space of $SO(2,1)$ Chern-Simons theory is Teichmüller space. In addition, Goldman [25] has shown that the symplectic form corresponding to the $SO(2,1)$ Chern-Simons Poisson bracket (2.19) is in fact a multiple of a natural Kähler form on Teichmüller space called the Weil-Petersson form (denoted ω_{WP}). The exact relation between the two is [26][15]:

$$\frac{1}{2\pi\hbar}\omega = \frac{k}{4\pi^2}\omega_{WP}. \tag{4.33}$$

As we will see in the next section, ω_{WP} is an exact form on Teichmüller space, so the phase space of $SO(2,1)$ is trivially quantizable. Since ω_{WP} is a Kähler form, we can apply the entire process of Kähler quantization and end up with a quantum Hilbert space that is the space of holomorphic sections of a trivial bundle over $\mathcal{T}_g$. For the $SO(2,1) \times SO(2,1)$ Chern-Simons theory that is locally equivalent to 3d gravity, we simply add our two Weil-Petersson forms (with coefficients $\propto k_L$ and $-k_R$) to get a Kähler form on $\mathcal{T}_g \times \mathcal{T}_g$ and proceed exactly as before, ending up again with the space of holomorphic sections of a trivial line bundle. As we have already seen, however, 3d gravity is not globally a Chern-Simons theory, and the above phase space must be quotiented by the mapping class group. Over the whole phase space we unfortunately have very little idea what this does to our symplectic geometry—whether our symplectic structure remains Kähler under the identifications of the quotient and what sort of quantizability conditions might be necessary for the existence of a prequantum line bundle.

We can make some headway by reducing the phase space to that of our minisuperspace model of static moduli solutions. On this slice of phase space the theory reduces to a single $SO(2,1)$ Chern-Simons theory with coupling constant $k' = k_L - k_R$, so for $k' \neq 0$ the induced symplectic form is again a Kähler form related to the Weil-Petersson form by (4.33). We also have a better understanding of the action of the mapping class group on this slice, since it acts internally, taking our Chern-Simons phase space $\mathcal{T}_g$ to moduli space $\mathcal{M}_g$. In addition, we will see that the Weil-Petersson form is invariant under the action of the mapping class group on Teichmüller space, and so descends to a Kähler form on $\mathcal{M}_g$. Consequently, if moduli space is quantizable, we can again apply our Kähler polarization procedure to determine the quantum Hilbert space.

The impediment to accomplishing this is that we must first find a prequantum line bundle, and this will only exist when $(k'/4\pi^2)\omega_{WP}$ is in integer cohomology on $\mathcal{M}_g$. This requirement is nontrivial, since ω_{WP} is no longer a globally exact form, and the quotient by Γ has left us with a topologically complicated space that may contain noncontractible two-cycles. In addition to this, we must alter what we mean by 'integer cohomology' to account for the fact that $\mathcal{M}_g$ is an orbifold. Because of this, there are closed loops on $\mathcal{M}_g$—the ones that pass through orbifold points—which are not boundaries of any smooth surface embedded in $\mathcal{M}_g$. However, they *will* always be boundaries of embeddings of surfaces which are themselves orbifolds, and whose orbifold points are mapped to orbifold points in $\mathcal{M}_g$. Repeating the arguments of section 4.2, we see that the new requirement for the wave function to be single-valued around these loops is that $(k'/4\pi^2)\omega_{WP}$ must also integrate to an integer over closed two-cycles with orbifold points at orbifold points on $\mathcal{M}_g$. To understand these conditions, we need a better understanding of the geometry of moduli space and the properties of the Weil-Petersson form. We now turn to an exposition on these topics.

Chapter 5

The geometry of moduli space

We have introduced the moduli space $\mathcal{M}_g$ as the space of hyperbolic (constant-negative-curvature) metrics on a given genus g Riemann surface Σ_g. For the purposes of this discussion we will actually want to consider the more general case of a genus g Riemann surface with n marked points, or punctures, which will become singularities in the hyperbolic metric on the surface. We will denote this surface $\Sigma_{g,n}$. For this more general case the moduli space will be denoted $\mathcal{M}_{g,n}$. This space can actually be constructed in a few different ways. One, as we have already seen, is as the space of embeddings of the fundamental group of $\Sigma_{g,n}$ into $PSL(2, R)$. Another is as the space of conformal classes of metrics on $\Sigma_{g,n}$, since by an extension of the uniformization theorem from section 2.2, every smooth metric on $\Sigma_{g,n}$ is conformally equivalent to a hyperbolic metric. Still another is as the space of genus-g algebraic curves with n marked points. Regardless of all these constructions, we will find it most useful to think of $\mathcal{M}_{g,n}$ in terms of our original definition, as the space of hyperbolic metrics on $\Sigma_{g,n}$.

The first three sections of this chapter present elements of the vast and well-studied theory of moduli spaces which will be relevant to our analysis.

Many of the results in these sections are taken from lecture notes by Harer [27] and the later review by Do [28], which the reader may wish to consult for a more in-depth examination of moduli spaces and their Weil-Petersson volumes.

5.1 Teichmüller space

Just as we have already alluded to for the $n = 0$ case, $\mathcal{M}_{g,n}$ will be constructed as a quotient of a generalized Teichmüller space, denoted $\mathcal{T}_{g,n}$, by the action of the mapping class group $\Gamma_{g,n}$ of our surface. We will start, therefore, by giving an explicit construction of this Teichmüller space. To do so, we need to answer the question: Exactly what information to we need in order to completely specify a hyperbolic metric with n singularities on a genus-g Riemann surface?

Our strategy will be to first answer this question for a very simple surface, and then use this surface as the building block with which to construct more complicated surfaces. The building block we use is a sphere with three disks removed: a surface often referred to as a *pair of pants*. Consider a pair of pants with boundary curves ('cuffs') of lengths l_1, l_2 and l_3. Our founding observation is the fact that if we demand that all three boundary curves be geodesics, then specifying the lengths (l_1, l_2, l_3) completely determines a hyperbolic metric on the pair of pants. To see this, consider a right-angled hexagonal domain on the hyperbolic disk, whose boundaries are all geodesic (Fig. 5.1). Specify the lengths of every second boundary geodesic to be $\frac{1}{2}l_1$, $\frac{1}{2}l_2$ and $\frac{1}{2}l_3$. The lengths of the geodesics in between are completely determined by the requirement that they intersect at right angles, and this completely determines the shape of the hexagonal domain. We now take two congruent copies of this hexagonal domain and glue them together along the 'in between' geodesics. The result

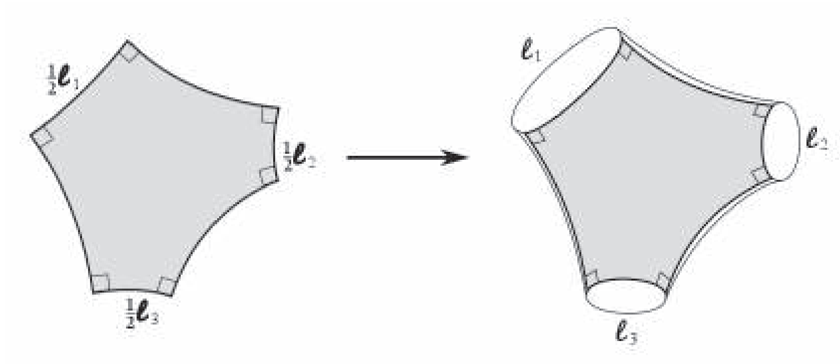

Figure 5.1: A hyperbolic hexagon is doubled and glued to construct a pair of pants.

is a pair of pants with metric completely determined by the metric on the hyperbolic disc, and cuff lengths (l_1, l_2, l_3). The in between geodesics become a set of three geodesics on the pair of pants which each intersect two geodesic cuffs at right angles. These are called the *seams* of the pair of pants.

The next observation is that every surface $\Sigma_{g,n}$ can be cut along $3g-3+n$ closed curves into a disjoint union of $2g-2+n$ pairs of pants. Here we consider the n punctures to be boundary geodesics of zero length, so when performing this decomposition each puncture becomes a cuff on a pair of pants. Due to a theorem that states that every closed curve on a hyperbolic surface is homotopic to a unique closed geodesic, it is always possible to perform this cutting into pairs of pants along geodesic curves. Therefore after cutting we are left with a set of pairs of pants with geodesic boundary components. Such a decomposition of our surface is called a pants decomposition. This decomposition is far from unique, which we can easily see just by examining the case of a genus-2 handle body (Fig. 5.2).

Since the hyperbolic metric on each pair of pants is completely determined by the lengths of its boundary geodesics, specifying the lengths of the $3g-3+n$ cut geodesics will fix the metric on every pair-of-pants building block of $\Sigma_{g,n}$. However, this is not yet quite the same as determining the hyperbolic metric

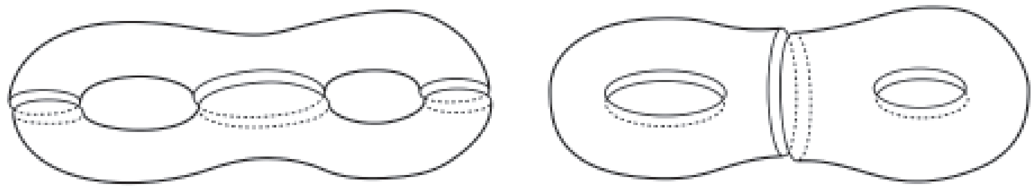

Figure 5.2: Two pants decompositions of a genus-2 Riemann surface

over the whole surface, because we have yet to specify how pairs of pants are glued together along their shared geodesics. The ambiguity lies in the fact that, for any gluing of two corresponding cuffs, we could just as easily cut them apart again, rotate one cuff with respect to the other, and glue them back together. Since we are gluing along geodesics, this new gluing will still result in a smooth hyperbolic surface, but with a different hyperbolic metric that reflects the twist. To parametrize the twists, we choose a disjoint set of closed curves C on $\Sigma_{g,n}$, all of whose elements transversely intersect the pants decomposition, and which reduces to three disjoint arcs on each pair of pants. As our 'zero twist' gluing, we choose the gluing for which each element of C is homotopic to a closed geodesic which is a union of seams on the pairs of pants. For any other gluing, the twist parameter τ for a given cut geodesic γ is the (signed) geodesic distance along γ between the endpoints of the two seams that formed a single geodesic on the zero-twist gluing (Fig. 5.3). We allow τ can take any value on $\mathbb{R}$, although it is clear that for a geodesic of length l, the

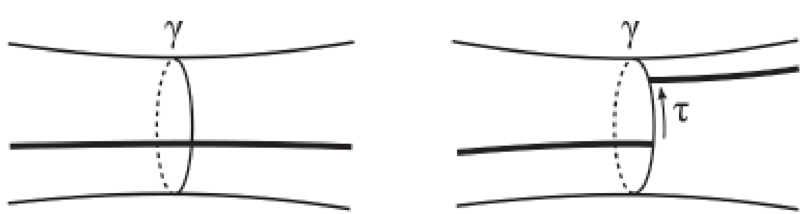

Figure 5.3: Two seams form a single geodesic on an untwisted gluing (left) but are separated by a distance τ on a twisted gluing (right).

twist parameters τ and $\tau+l$ result in exactly the same gluing, and therefore the same hyperbolic surface. The operation of cutting, twisting by l and re-gluing is called a Dehn twist, and is an example of a large diffeomorphism on $\Sigma_{g,n}$. Our eventual quotient by the mapping class group will identify points related by Dehn twists, but in Teichmüller space they are treated as distinct since the map between them is not an infinitesimally generated diffeomorphism of the surface.

The set of coordinates $(l_k, \tau_k), k = 1, \ldots, 3g-3+n$ are called the Fenchel-Nielsen coordinates, and are global coordinates on Teichmüller space. Each choice of a set of values for these coordinates uniquely labels a hyperbolic metric on $\Sigma_{g,n}$ that is distinct up to infinitesimally generated diffeomorphisms. Note that these coordinates are not unique; there is a set of them for each distinct pants decomposition of $\Sigma_{g,n}$, and in general the transformation between two sets of Fenchel-Nielsen coordinates does not have a closed form. Each l_k takes values along the positive real line and each τ_k takes values along the entire real line, so Teichmüller space is a $(6g-6+2n)$-dimensional manifold isomorphic to $(\mathbb{R}_+ \times \mathbb{R})^{3g-g+n}$, the tensor product of $3g-3+n$ upper-half planes. Alternately, we can think of each pair (l_k, τ_k) as defining set of polar coordinates on an infinite-sheeted cover of the punctured 2d plane, with radial coordinate $r = l_k$ and an unwrapped polar angle $\phi = 2\pi\tau_k/l_k$ that takes values in $\mathbb{R}$. This gives us a natural way of thinking about the Dehn twists: they are rotations by 2π that take us from a given point to the same point on a different sheet of the cover. $\mathcal{T}_{g,n}$ is then the tensor product of $3g-3+n$ such surfaces.

We will also find it necessary to consider the compactification of Teichmüller space that arises by allowing one or several geodesics on $\Sigma_{g,n}$ to collapse to zero size, producing two identified cusps on the surface. The space that results from adding these points to Teichmüller space is called the Delign-Mumford compactification, denoted $\overline{\mathcal{T}}_{g,n}$. This compactification has the mathematically advantageous property of turning Teichmüller space into an algebraic space. To determine what points are actually added, there are two situations we must consider for any given pants decomposition. One is the case where the collapsed geodesic is one of the cut geodesics in the decomposition. In this case, the point that must be added to Teichmüller space is just the point $l_k = 0$. Note that this is in fact a single point, since the definition of τ breaks down for vanishing geodesics. This is intuitive in the picture where (l_k, τ_k) parametrizes an infinite cover of the punctured plane, since adding the vanishing-geodesic surface to Teichmüller space simply means filling in the puncture. On the other hand, if the vanishing geodesic is not one of those cut by the pants decomposition, it corresponds to adding a point in a limiting case where at least two of l_k values (and possibly more) simultaneously go to infinity. What adding all these points at infinity does to the overall topology of the space is much less intuitively clear. The overall set $\overline{\mathcal{T}}_{g,n} - \mathcal{T}_{g,n}$ of added points is called the compactification locus.

5.2 The Weil-Petersson Form

In a given set of Fenchel-Nielsen coordinates, the Weil-Petersson form, which we said earlier was a multiple of the symplectic form of our Chern-Simons theory, is given by

$$\omega_{WP} = dl^k \wedge d\tau_k. \tag{5.1}$$

While this definition would at first glance seem to depend on our choice of pants decomposition for $\Sigma_{g,n}$, Wolpert [29] has shown that in fact ω_{WP} takes this form for *any* choice of Fenchel-Nielsen coordinates.

The Weil-Petersson form is clearly compatible with the complex structure defined by $J(\partial/\partial l_k) = \partial/\partial\tau_k$, $J(\partial/\partial\tau_k) = -\partial/\partial l_k$, and is in fact a Kähler form on $\mathcal{T}_{g,n}$. Expressed in these coordinates, ω_{WP} is clearly invariant under the flows generated by $\partial/\partial l_k$ and $\partial/\partial\tau_k$. In particular, it is invariant under the Dehn twist operation $\tau_k \to \tau_k + l_k$.

As a symplectic form, ω_{WP} can be used to define a volume form on $\mathcal{T}_{g,n}$ given by

$$\frac{\omega^{3g-3+n}}{(3g-3+n)!} = dl_1 \wedge d\tau_1 \wedge dl_2 \wedge d\tau_2 \wedge \ldots \wedge dl_{3g-3+n} \wedge d\tau_{3g-3+n}. \qquad (5.2)$$

Here and in the future we drop the 'WP' subscript whenever there is no confusion about what symplectic form we are denoting.

Wolpert [29] has shown that the Weil-Petersson form extends to a symplectic form on the Delign-Mumfold compactification $\overline{\mathcal{T}}_{g,n}$. Let us now consider its behaviour when confined to a certain subspace of the compactification locus $\overline{\mathcal{T}}_{g,n} - \mathcal{T}_{g,n}$. Consider two surfaces Σ_{g_1,n_1} and Σ_{g_2,n_2} which satisfy the conditions $g_1 + g_2 = g - m + 1$ and $n_1 + n_2 = n + 2m$ for some integer $m \leq n_1, n_2$. We can join m cusps on Σ_{g_1,n_1} to cusps on Σ_{g_2,n_2} (Fig. 5.4) to obtain $\Sigma_{g,n}$: a genus-g surface with n marked points and m double cusps (vanishing-length geodesics). We now choose a pants decomposition for each component surface. Denote the set of cut geodesics on the first surface by $\Gamma_1 = \{\gamma_i : i = 1, \ldots, 3g_1 - 3 + n_1\}$, the set of cut geodesics on the second surface by $\Gamma_2 = \{\gamma_j : j = 1, \ldots, 3g_2 - 3 + n_2\}$, and the set of zero-length geodesics around the double cusps on $\Sigma_{g,n}$ by $\Gamma_0 = \{\gamma_k : k = 1, \ldots, m\}$. Then $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ is a set of $3g - 3 + n$ geodesics that defines a perfectly good pants decomposition of $\Sigma_{g,n}$. We use this pants decomposition to define a set of coordinates on $\overline{\mathcal{T}}_{g,n}$. In these coordinates, the Weil-Petersson form is given

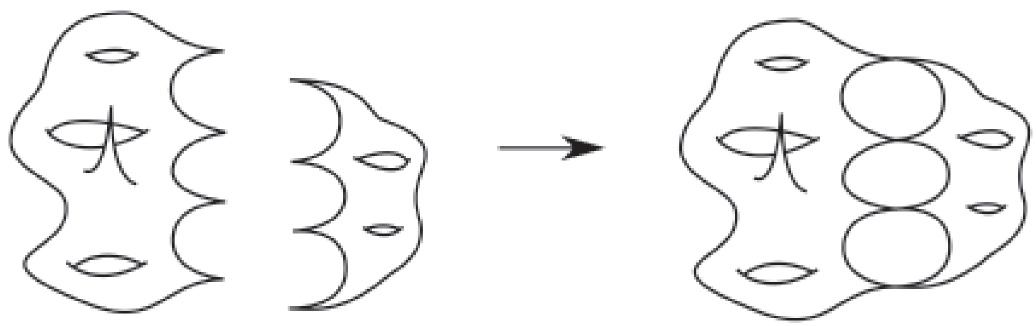

Figure 5.4: Two surfaces Σ_{g_1,n_1} and Σ_{g_2,n_2} join along cusps to form a single surface $\Sigma_{g,n}$.

by

$$\omega = dl^i \wedge d\tau_i + dl^j \wedge d\tau_j + dl^k \wedge d\tau_k, \tag{5.3}$$

where l_i, τ_i are the lengths and twists on the γ_i's, etc. Of course, for the surface we are considering all of our l_k values are fixed to 0. On the subspace whose points correspond to such surfaces (which is clearly isomorphic to $\overline{\mathcal{T}}_{g_1,n_1} \times \overline{\mathcal{T}}_{g_2,n_2}$), the Weil-Petersson form reduces to

$$\omega = dl^i \wedge d\tau_i + dl^j \wedge d\tau_j = \omega_1 + \omega_2 \tag{5.4}$$

where ω_1 and ω_2 are the Weil-Petersson forms on $\overline{\mathcal{T}}_{g_1,n_1}$ and $\overline{\mathcal{T}}_{g_2,n_2}$, respectively. So we have shown that ω reduces to the sum of the Weil-Petersson forms on the two component surfaces of $\Sigma_{g,n}$. This is called the restriction phenomenon [6]. It is easy to extend the above analysis to show that this same phenomenon occurs whenever $\Sigma_{g,n}$ is divisible along double cusps into an arbitrary number of lower-genus Riemann surfaces: the Weil-Petersson form reduces to a sum of Weil-Petersson forms on those surfaces.

5.3 Moduli space

To construct the moduli space $\mathcal{M}_{g,n}$, we quotient $\mathcal{T}_{g,n}$ by the action of the mapping class group $\Gamma_{g,n}$. As previously stated, $\Gamma_{g,n}$ is the group of all diffeomorphisms on $\Sigma_{g,n}$, modded out by those that are connected to the identity. Since the surfaces represented by points in Teichmüller space are already unique up to infinitesimal diffeomorphisms, this quotient ensures that the points in $\mathcal{M}_{g,n}$ are in one-to-one correspondence with truly unique hyperbolic metrics on $\Sigma_{g,n}$.

Dehn [30] showed that the mapping class group is completely generated by performing Dehn twists along geodesics on $\Sigma_{g,n}$. For a Dehn twist performed along one of the cut geodesics in our pants decomposition, our picture of the (l, τ) surface as an infinite cover of the plane gives us an easy way of visualizing the quotient of Teichmüller space by this element of $\Gamma_{g,n}$. The quotient simply identifies corresponding points on each leaf of the cover, reducing it down to a single punctured plane. So moduli space is the tensor product of $3g-3+n$ punctured planes quotiented by the subgroup of $\Gamma_{g,n}$ corresponding to twists around geodesics not in the pants decomposition. This quotient is unfortunately much harder to visualize. We do know, however, that the action of this group is properly discontinuous but not free, and therefore $\mathcal{M}_{g,n}$ is an orbifold. If we quotient $\overline{\mathcal{T}}_{g,n}$ instead of $\mathcal{T}_{g,n}$, we obtain $\overline{\mathcal{M}}_{g,n}$: the Deligne-Mumford compactification of moduli space. This is the space of unique hyperbolic metrics on $\Sigma_{g,n}$ that are allowed to have vanishing closed geodesics.

The Weil-Petersson form, being invariant under Dehn twists, is invariant under the action of the entire mapping class group. Therefore it descends to a Kähler form on $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$, giving them a Kähler orbifold structure. The volume form (5.2) also descends to a volume form on these spaces. We define the Weil-Petersson volume of moduli space by

$$V_{g,n} = \int_{\mathcal{M}_{g,n}} \frac{\omega^{3g-3+n}}{(3g-3+n)!} \tag{5.5}$$

Because the compactification locus $\mathcal{D} = \overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$ is a union of submanifolds of positive codimension [6], we can equivalently perform the integral over $\overline{\mathcal{M}}_{g,n}$ and the result will not change.

Mirzakhani [31], building on the intersection theory on moduli space developed by Witten [32] and Kontsevich [33], has found a recursion algorithm that allows one in principle to compute these volumes to arbitrarily high genus. In practice this algorithm becomes unwieldly very quickly as genus increases. Zograf [34], using an alternate algorithm, has gathered enough data to conjecture that at large g the Weil-Petersson volumes have the asymptotic behaviour

$$V_{g,n} = \frac{1}{\sqrt{g\pi}}(4\pi^2)^{2g-3+n}(2g-3+n)!\left[1 + c_n g^{-1} + O(g^{-2})\right] \quad \text{as} \quad g \to \infty. \tag{5.6}$$

This conjecture has not been proven, but is supported by empirical data [34] and analytical evidence by Mirzakhani [35].

On the other hand, for small genus many exact results are known. The ones we will need here are $V_{1,1} = \frac{\pi}{6}$ and $V_{0,4} = \frac{\pi}{3}$, originally calculated by Wolpert [36][6]. These will be important enough to our result that we should understand how they are derived.

5.4 Volumes of two-dimensional moduli spaces

Let us now examine those moduli spaces which are two-dimensional. These will be important to us due to the property that their Weil-Petersson forms *are* volume forms, so their volumes are simply $\int_{\mathcal{M}} \omega$. Since $d = 6g-6+2n$, we see that there are only two moduli spaces for which $d = 2$: $\mathcal{M}_{1,1}$, the moduli space of the once-punctured torus $\Sigma_{1,1}$; and $\mathcal{M}_{4,0}$, the moduli space of the four-punctured sphere $\Sigma_{0,4}$. Both $\Sigma_{1,1}$ and $\Sigma_{0,4}$ can be pants-decomposed by cutting along a single geodesic, so their Teichmüller spaces are identical. In addition their mapping class groups are the same, each being generated by Dehn twists around the two geodesics that make up their abelian fundamental groups. Given this, it is not surprising that in fact $\mathcal{M}_{1,1} \approx \mathcal{M}_{0,4}$ [6]. The map between the two proceeds by noting that every once-punctured torus has a four-punctured torus as a fourfold cover, and every four-punctured torus is a twofold cover of a four-punctured sphere. This defines a map from any given point on $\mathcal{M}_{1,1}$ to a point on $\mathcal{M}_{0,4}$ and vice verse, and the multiplicity of the covers gives us the relation $\omega_{0,4} = 2\omega_{1,1}$[6]. So we are guaranteed that $V_{0,4} = 2V_{1,1}$, and are left with only one Weil-Petersson volume to find.

Although Wolpert [36] was the first to calculate this volume, we will derive it using a simpler method due to Mirzakhani [37]. We first define the following cover of $\mathcal{M}_{1,1}$:

$$\mathcal{M}^*_{1,1} = \{(X, \gamma) : X \in \mathcal{M}_{1,1}, \gamma \text{ is a simple closed geodesic on } X.\} \tag{5.7}$$

The projection $\pi : \mathcal{M}^*_{1,1} \to \mathcal{M}_{1,1}$ is defined by forgetting about the geodesic: $\pi(X, \gamma) = X$. Denote by $\pi^*\omega$ the pullback of the Weil-Petersson form onto $\mathcal{M}^*_{1,1}$. Now, consider a function $f : \mathcal{M}^*_{1,1} \to \mathbb{R}$. We can define the pushforward $\pi_* f : \mathcal{M}_{1,1} \to \mathbb{R}$ by [37]

$$\pi_* f(X) = \sum_{Y \in \pi^{-1}(X)} f(Y). \tag{5.8}$$

We then have a relation between the integrals of these two functions:

$$\int_{\mathcal{M}_{1,1}} (\pi_* f)\omega = \int_{\mathcal{M}^*_{1,1}} (\pi^*\omega) f \tag{5.9}$$

Our strategy will be to find a function f for which $\pi_* f = 1$. This equation will then give us an integral over the covering space that is equal to the Weil-Petersson volume of $\mathcal{M}_{1,1}$.

We now wish to show that [37]

$$\mathcal{M}^*_{1,1} \cong \{(l, \tau) : l \in \mathbb{R}_+, 0 \leq \tau \leq l\}/ \sim \tag{5.10}$$

where $(l, 0) \sim (l, l)$ for all $l \in \mathbb{R}_+$. This is the space obtained by quotienting $\mathcal{T}_{1,1}$ by a single Dehn twist, and we will denote it $\mathcal{T}_{1,1}/ \sim$. We show the congruence of these spaces by finding a bijective map between them. The injection from $\mathcal{M}^*_{1,1}$ to $\mathcal{T}_{1,1}/ \sim$ is straightforward: (X, γ) simply maps to the point $(l, \tau) = (l(\gamma), \tau(\gamma))$, where $l(\gamma)$ and $\tau(\gamma)$ are the length and twist parameters of the geodesic γ. The domain restriction $0 \leq \tau \leq l$ ensures that this map is injective. In the reverse direction, any pair (l_0, τ_0) defines a point on $\mathcal{T}_{1,1}$, and the projection from $\mathcal{T}_{1,1}$ to $\mathcal{M}_{1,1}$ maps this to a unique point X on $\mathcal{M}_{1,1}$. If X is not an orbifold point we are guaranteed to have a unique geodesic γ on X for which $(l(\gamma), \tau(\gamma)) = (l_0, \tau_0)$, so this pair also determines a unique geodesic $\gamma(l_0, \tau_0)$. If X *is* an orbifold point, we may have multiple geodesics with the same length and twist parameters. In this case we simply have to choose some systemic way of picking one of them, such as taking $\gamma(l_0, \tau_0) \equiv \lim_{\epsilon \to 0^+} \gamma(l_0 - \epsilon, \tau_0 - \epsilon)$. This completes the second injective map. Defined like this our two injective maps are inverses of each other, and so define a bijective map between our two spaces which are therefore equivalent.

Since $\mathcal{M}^*_{1,1}$ is a quotient of $\mathcal{T}_{1,1}$ and our (l, τ) coordinates are simply Fenchel-

Nielsen coordinates, the pullback of the Weil-Petersson form onto $\mathcal{M}^*_{1,1}$ takes the usual form $dl \wedge d\tau$. The final piece of the puzzle is an identity proven by McShane [38], which states that on a hyperbolic torus X with a single cusp,

$$\sum_{\gamma} \frac{2}{1+\exp(l(\gamma))} = 1 \tag{5.11}$$

where the sum runs over all simple closed geodesics on X. With this we see that if we take $f(l) = \frac{2}{1+\exp(l)}$, we have $\pi_* f = 1$ and equation (5.9) becomes [37]

$$\int_{\mathcal{M}_{1,1}} \omega = \int_{\mathcal{M}^*_{1,1}} f(l) dl \wedge d\tau = \int_0^\infty \int_0^l \frac{2}{1+\exp(l)} d\tau dl = \frac{\pi^2}{6} \tag{5.12}$$

So as previously stated, $V_{1,1} = \frac{\pi^2}{6}$ and $V_{0,4} = 2V_{1,1} = \frac{\pi^2}{3}$.

5.5 Cohomology of the Weil-Petersson form

We are now in a position to calculate the cohomology of ω on our actual moduli space $\mathcal{M}_g$ of gravity solutions. To do this, we first need to understand the second homology group on this space; i.e. what closed two-cycles exist over which we can integrate ω. We will refer to the *manifold* homology group if we are considering only cycles which are embeddings of two-dimensional manifolds into $\mathcal{M}_g$, and denote this group $H^*_2(\mathcal{M}_g, \mathbb{Q})$ (where $\mathbb{Q}$ denotes that, as an orbifold, $\mathcal{M}_g$ has rational homology.) The *orbifold* homology group, where we are allowing cycles which are embeddings of orbifolds, will be denoted $H_2(\mathcal{M}_g, \mathbb{Q})$. It is this second homology group that will give us the quantization condition on the Chern-Simons coupling k'.

Every orbifold cycle must corresponding a manifold cycle obtained by projecting the orbifold's covering space down to the orbifold itself and then onto

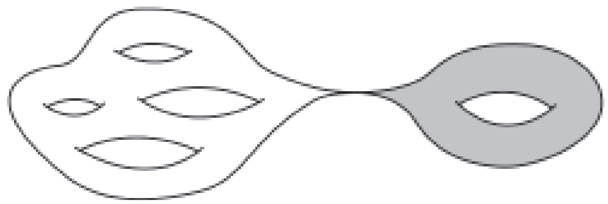

Figure 5.5: A once-punctured torus (shaded) and once-punctured genus $g-1$ surface are joined into a single punctureless surface. Allowing the torus to vary over its moduli space sweeps out an orbifold two-cycle in $\overline{\mathcal{M}}_g$.

the cycle, sweeping it out multiple times. So the generators of these two homology groups should be in one-to-one correspondence. Harer [39] has shown that $H_2^*(\mathcal{M}_g, \mathbb{Q})$ is generated by a single element. By our previous considerations, this means that our orbifold homology group $H_2(\mathcal{M}_g, \mathbb{Q})$ must also be one-dimensional. It is not clear how to define an actual two-cycles on $\mathcal{M}_g$ that might generate its second homology group. On the other hand, Wolpert [6] has shown that for $g \geq 3$ the dimension of the manifold homology group $H_2^*(\overline{\mathcal{M}}_g, \mathbb{Q})$ is $2 + \lfloor g/2 \rfloor$, and has also found an explicit construction of its cycles. We will repeat his analysis here, slightly modified to directly produce the generators of the orbifold homology group.

The $2 + \lfloor g/2 \rfloor$ homology classes in $H_2(\overline{\mathcal{M}}_g; \mathbb{Q})$ can be constructed [6] as maps from covers of $\overline{\mathcal{M}}_{1,1}$ and $\overline{\mathcal{M}}_{0,4}$ onto $\overline{\mathcal{M}}_g$. To embed $\overline{\mathcal{M}}_{1,1}$, we start with a genus $g-1$ surface $\Sigma_{g-1,1}$ with a single puncture. We attach the cusp on this surface to the cusp on a once-punctured torus (Figure 5.5), creating a surface Σ_g corresponding to some point on the compactification locus $\mathcal{D} = \overline{\mathcal{M}}_g - \mathcal{M}_g$. We then allow the punctured torus to vary over its entire moduli space while keeping the moduli on $\Sigma_{g-1,1}$ fixed, naturally sweeping out a two-cycle which is the image of $\overline{\mathcal{M}}_{1,1}$. Over this two-cycle the restriction phenomenon reduces ω to $\omega_{1,1}$, the Weil-Petersson form of the once-punctured torus. So integrating ω over this cycle simply gives us the Weil-Petersson volume $V_{1,1} = \frac{\pi^2}{6}$.

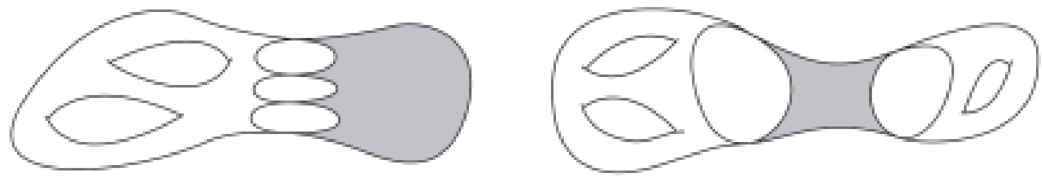

Figure 5.6: A four-punctured sphere (shaded) is connected to a) one four-punctured genus $g-3$ surface, or b) two two-punctured surfaces with genuses adding to $g-2$. Varying the sphere over its moduli space sweeps out a two-cycle for each configuration.

All such cycles can be deformed into one another by varying the moduli of $\Sigma_{g-1,1}$ in the above construction, so the above provides us with only a single homology class. We therefore still have $1 + \lfloor g/2 \rfloor$ homology classes to find. These will all be defined as embeddings of a sixfold cover of $\overline{\mathcal{M}}_{0,4}$ into $\overline{\mathcal{M}}_g$, and can be labeled by an integer m, $m = 0, \ldots, \lfloor g/2 \rfloor$. We proceed similarly to the construction of the first cycle. For $m = 0$, we start with a genus $g-3$ surface with four punctures and attach its cusps to those of a four-punctured sphere (Figure 5.6a). For $m = 1, ..., \lfloor g/2 \rfloor$, we instead start with a two two-punctured surfaces of genuses $m-1$ and $g-m-1$ respectively, and attach both by their cusps to a single four-punctured sphere (Figure 5.6b). As with the torus, the punctured sphere is then allowed to vary over its entire moduli space while the other moduli are kept fixed. This defines the remaining $1 + \lfloor g/2 \rfloor$ nontrivial two-cycles. These cycles are embeddings of the six-fold cover of $\overline{\mathcal{M}}_{0,4}$ corresponding to the moduli space of a 4-punctured sphere with labelled points [6]; in this case the punctures are labelled by their point of attachment to the other surfaces. They are in fact manifold cycles, each being topologically a three-punctured sphere (see, e.g. [40]). The restriction phenomenon again reduces ω to $\omega_{0,4}$ on these cycles, and since they are sixfold covers of $\overline{\mathcal{M}}_{0,4}$ the integral over each of them evaluates to $6V_{0,4} = 2\pi^2$.

Since the smallest integral of ω over a two-cycle is $\frac{\pi^2}{6}$, we see that the smallest multiple of the Weil-Petersson form which is in integer cohomology on $\overline{\mathcal{M}}_g$ is $(6/\pi^2)\omega$. For any integer multiple of this form there will exist a line bundle on $\overline{\mathcal{M}}_g$ with a compatible connection whose curvature is this form. Despite our initial fears, the prefactor on ω does not depend on genus for $g > 3$, but is a constant. Therefore these line bundles can be defined on moduli spaces of arbitrarily high genus with an unchanging choice of prefactor.

What about second homology group, and cohomology of ω, on the uncompactified moduli space $\mathcal{M}_g$? All of the cycles defined above exist on the compactification locus $\mathcal{D}$, so none of them alone is a suitable candidate for the generator of $H_2(\mathcal{M}_g;\mathbb{Q})$. However, since any cycle in $\mathcal{M}_g$ is a cycle in $\overline{\mathcal{M}}_g$, there must exist some combination of the elements of $H_2(\overline{\mathcal{M}}_g;\mathbb{Q})$ which can be smoothly deformed off of $\mathcal{D}$ to form the generator of this group. Let us call the resulting two-cycle ξ. The only question that remains is whether or not ω interacts nontrivially with ξ; i.e. whether it integrates to a nonzero number. Wolpert [6] has proven that this is indeed the case by showing that $H_{6g-8}(\overline{\mathcal{M}}_g;\mathbb{Q})$ can be generated by $[\omega]$ (the Poincaré dual of ω) and the disjoint elements of $\mathcal{D}$. Therefore since ξ is both topologically nontrivial and does not intersect with $\mathcal{D}$, it must intersect with $[\omega]$. Thus, we are guaranteed that the integral of ω over ξ is $\frac{\pi^2}{6}n$ for some integer n, and $(6/\pi^2)\omega$ is again in integer cohomology. Here $6/\pi^2$ is no longer guaranteed to be the smallest prefactor: if n is a multiple of 2 or 3 smaller values will be allowed. n may in fact vary in some nontrivial way with g, allowing smaller prefactors for some genuses than for others. So given our current knowledge, $(6/\pi^2)\omega$ is the still smallest multiple of ω that we can guarantee to be in integer cohomology, and so have an associated line bundle, at arbitrarily high genus.

Chapter 6

Results and Outlook

6.1 The quantization condition

Recall that our quantization condition was that $\frac{k'}{4\pi^2}\omega_{WP}$ must be in integer cohomology for our minisuperspace model to be quantizable. Our initial worry was that this quantization condition would force k' to diverge at high genus, effectively eliminating parity-violating 3d gravity as a self-consistent theory. We now see, however, that we are assured of quantizability at arbitrarily high genus if $k' \in 24\,\mathbb{Z}$.

If we allow our theory of gravity to include classical space-times whose spatial slices have double-cusp singularities, the phase space of our minisuperspace model becomes $\overline{\mathcal{M}}_g$ and we a guaranteed that *only* k' values which are multiples of 24 will lead to quantizable theories. This is a much stronger quantum condition than any of those considered by Witten in the context of AdS/CFT [18] or than is necessary for us to be able to Wick-rotate the theory to Euclidean signature, for which k' only has to be integer. (Although, conversely, these contexts also provide a quantization condition for k, the prefactor of the classical Einstein term, about which we have nothing to say.)

However, there may be good reasons for us *not* to include double-point singular spaces in our phase space and work only with $\mathcal{M}_g$, in which case it is possible that the cohomology of the Weil-Petersson form allows for smaller values of k'. It would be interesting if simply including the limiting points in the phase space is enough to alter the quantum condition on the entire theory, and ideally we would like to understand whether this is the case by explicitly working out the cohomology of ω on $\mathcal{M}_g$. However, the most likely scenario seems to be that if this cohomology has any dependence at all on g, $k' \in 24\,\mathbb{Z}$ will remain the only choice which remains valid for all genus, as required for a consistent gravity theory.

We have obtained this result on a reduced theory that lives on a half-dimensional slice of the full phase space, and one might doubt whether this result has any bearing on the quantization of the full theory. It is easy to see, however, that the quantization condition for the reduced phase space is at least a necessary condition for the quantization of the full theory. This is because ξ, our one nontrivial two-cycle on $\mathcal{M}_g$, is also a nontrivial two-cycle on $\mathcal{T}^*\mathcal{M}_g$. Furthermore, the integral of the reduced symplectic structure (the usual multiple of ω_{WP}) over a cycle confined to $\mathcal{M}_g$ is by definition the same as the integral of the full symplectic structure ω over the same cycle. Therefore whatever other nontrivial two-cycles may be present in the full phase space, the integral $\int_\xi \omega$ imposes the exact same quantization condition on the full theory as on the reduced theory.

6.2 Semiclassical behaviour of the Hilbert space

We saw in equation (2.20) that the commutator is inversely proportional to k'. Therefore the large k' limit is a semiclassical limit where the commutators go to zero. Since k' can be an arbitrary multiple of 24 there is no obstruction to us taking this limit. If we denote by L a valid prequantum line bundle for the symplectic manifold for $k' = 24$, then L^m provides a valid prequantization for $k' = 24m$, where m here is an arbitrary integer. Applying the standard prequantization scheme for a Kähler manifold, the quantum Hilbert space $\mathcal{H}$ for arbitrary m is the space of holomorphic sections of L^m. It is easy to check that ω_{WP} is a positive Kähler form, and therefore we can use the formula (4.32) for the dimension of the Hilbert space at large m. Plugging in the relation (4.33) for our symplectic form, we obtain

$$\dim\mathcal{H} = \dim H^0(\mathcal{M}_g, L^m) \to \left(\frac{6m}{\pi^2}\right)^{3g-3} \int_{\mathcal{M}_g} \frac{{\omega_{WP}}^{3g-3}}{(3g-3)!} \quad \text{as} \quad m \to \infty. \tag{6.1}$$

We recognize the integral in this expression as V_g, the Weil-Petersson volume of $\mathcal{M}_g$. Let us now see how this expression behaves when we *also* take the limit where g becomes large. Assuming Zograf's conjecture (5.6) for the limiting behaviour of V_g is true, we arrive at the expression

$$\dim\mathcal{H}_g \to \frac{(24m)^{3g-3}(2g-3)!}{(4\pi^2)^g\sqrt{g\pi}} \left[1 + cg^{-1} + O(g^{-2})\right] \quad \text{as} \quad g \to \infty. \tag{6.2}$$

This expression grows very quickly as g increases due to both the factorial and the exponential function of an already-large base. This is interesting if we consider the perspective that in a true theory of gravity, the full quantum Hilbert space of the theory should be the direct sum $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots$ of the individual Hilbert spaces at each genus. If the limiting behaviour of our minisuperspace model can be trusted to reflect the limiting behaviour of the full theory, the divergence of $\dim\mathcal{H}_g$ at large g in the classical limit implies that effectively all the states in the classical theory are off at infinite genus, and therefore the 'typical' state of the theory has an infinite-dimensional handle-body as its spatial slice. Thus it seems we may have a realization of the old idea of 'quantum foam', where spacetime at the quantum level is made up of a sea of tiny nontrivial topological connections.

It is possible that the actual divergence of our Hilbert space dimension is not actually a behaviour of the theory, but rather an artifact of the order in which we took our limits. The formula (4.32) used to take the first limit is only valid when all the higher cohomology groups $H^i(M, L^m)$ of the line bundle vanish [22], and while this is always true in the strict $m \to \infty$ limit it is possible that for any fixed m there is a maximum g value for which this approximation is valid. Above this cutoff we might hope that the dimension of $\mathcal{H}_g$ eventually falls off again, so that in the end we retain a finite-dimensional Hilbert space. However, it seems entirely likely that this divergent behaviour is completely robust; after all, Zograf's conjecture suggests, and Mirzakhani has in fact proven [35] that in the large g limit we have $\frac{V_g}{V_{g-1}} = 64\pi^4 g^2 + O(g)$. It seems rather odd to suggest that in any limit the larger phase space will contain less quantum states than the smaller one.

Thus we may have a real divergence. This is not unheard of: in ref. [41], Witten encounters a similar infrared divergence in the partition function over compact $\Sigma \times \mathbb{R}$ universes for $\Lambda = 0$ parity-conserving Einstein gravity in (2+1) dimensions, and argues that this is because the scale invariance of the theory allows for arbitrarily large space-times. If this is the source of the divergence in our theory, we might hope that it should disappear if we introduce a length scale—say, by considering surfaces that have a single boundary of fixed length A. These are exactly the sort of surfaces that might appear as states behind the event horizon of the BTZ black hole. The BTZ black hole is a topological black hole in $\Lambda < 0$ 3d gravity that can be constructed as a $\Sigma \times \mathbb{R}$ universe where Σ is cylinder, and the event horizon is the single closed geodesic on the cylinder [42]. This black hole has the usual horizon entropy, given by the event horizon length L in Planck units. So we should certainly hope that the Hilbert space of states behind the event horizon is not only finite-dimensional, but has dimension $\propto \exp(L)$ in the semiclassical limit. The moduli space of surfaces with a fixed boundary of length[1] L can be constructed in exactly the same way as those we have already discussed, and their Weil-Petersson volumes, denoted $V_{g,1}(L)$, can also be calculated using Mirzakhani's recursion relation. For low genus these have been calculated explicitly. The first three are [28]:

$$V_{1,1}(L) = \tfrac{1}{24}L^2 + \tfrac{\pi^2}{6} \tag{6.3}$$

$$V_{2,1}(L) = \tfrac{1}{442368}L^8 + \tfrac{29\pi^2}{138240}L^6 + \tfrac{139\pi^4}{23040}L^4 + \tfrac{169\pi^2}{2880}L^2 + \tfrac{29\pi^8}{192} \tag{6.4}$$

$$\begin{aligned} V_{3,1}(L) = {} & \tfrac{1}{53508833280}L^{14} + \tfrac{77\pi^2}{9555148800}L^{12} + \tfrac{3781\pi^4}{2786918400}L^{10} + \tfrac{47209\pi^6}{418037760}L^8 \\ & + \tfrac{127189\pi^8}{26127360}L^6 + \tfrac{8983379\pi^{10}}{87091200}L^4 + \tfrac{8497697\pi^{12}}{9331200}L^2 + \tfrac{9292841\pi^{14}}{4082400} \end{aligned} \tag{6.5}$$

Let us consider L large, compared both to the Planck scale and to l. This is the case where the infrared divergence should be the worst. In this regime, it should be the first term in each of these volumes that dominates the sum. So if the denominators of the first terms grow faster than exponentially, these volumes should eventually start to decrease for any arbitrarily large (but finite) value of L. Fortunately these first denominators are the easiest to calculate with Mirzakhani's formula. Figure 6.1 is a plot of the $(6g-4)^{\text{th}}$ roots of the first-term denominators for $g = 1 \ldots 500$, calculated using a Maple implementation of Mirzakhani's algorithm written by Liu and Xu. It is clear from the figure that these denominators are growing faster than exponential. Thus this

[1]Here we are using units where the AdS scale is $l = 1/\sqrt{-\Lambda} = 1$. The proper unitless quantity is L/l.

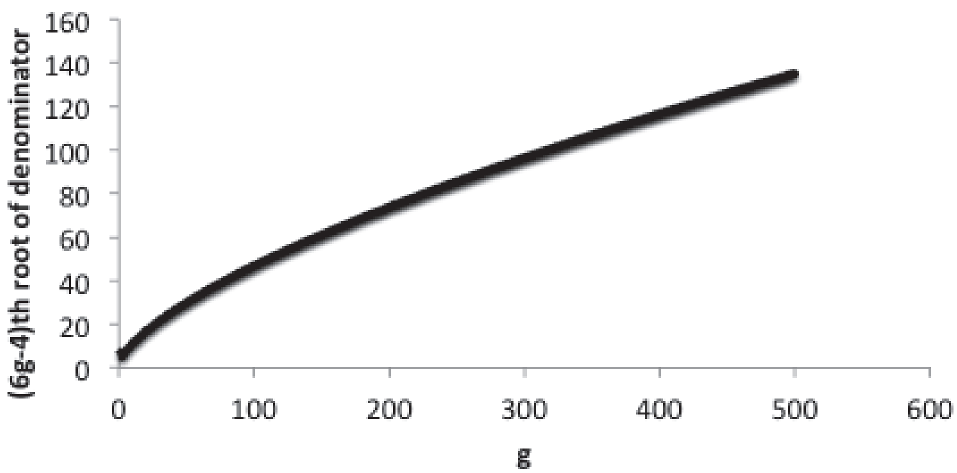

Figure 6.1: Plot of the $(6g-4)^{\text{th}}$ roots of the denominators of the first terms of $V_{g,1}(L)$ as a function of g. The y value at each genus is the smallest value that L can take such that the first term in $V_{g,1}(L)$ remains greater than 1.

first term at least provides no impediment to the hope that the Weil-Petersson volumes are well-behaved at large genus, and that the overall dimension of the Hilbert space will be finite. Of course, the reality is that as these terms vanish at higher genus, the next-order terms will begin to dominate, and even if all terms containing L eventually vanish at high enough genus we are sure to be left with the last, L-independent term in each sum. These terms are the Weil-Petersson volumes for compact surfaces with a single marked point, and unfortunately diverge at large genus in the exact same way as the V_g's [35]. We can think of the surface becoming so large and complicated behind the event horizon that it effectively becomes blind to the existence of a boundary, and recovers its old scale invariance. Thus the *best* case scenario seems to be that at large genus, the divergence for this state-counting is at least as bad as for our cosmological model. To produce a finite entropy we need some further mechanism to suppress the high-genus states, the way the high energy of UV photons suppresses the ultraviolet catastrophe in the study of black body radiation. It is unclear, in the absence of a Hamiltonian, what such a mechanism might be.

6.3 Outlook

We have found evidence of a novel quantization condition for the parity-violating Chern-Simons coupling in 3d gravity. Our result suggests that 3d gravity with a negative cosmological constant, when treated as a Chern-Simons theory, can only be quantized when k' is a multiple of 24. To test this condition more rigorously we would like to explicitly understand the the single nontrivial two-cycle that generates $H^2(\mathcal{M}_g, \mathbb{Q})$ and evaluate the integral of ω_{WP} over it. Only then will we know for sure what our quantization condition is for the case where our phase space excludes solutions with double-cusp singularities. More ambitiously, we would like to leave our minisuperspace model behind and tackle the quantization of the entire phase space $\mathcal{T}^*\mathcal{M}_g$. In particular we would like to know whether there are nontrivial two-cycles other than ξ in $H_2(\mathcal{T}^*\mathcal{M}_g, \mathbb{Q})$, and whether the cohomology of ω over these cycles changes the quantization condition for k' or provides one for k. In our reduced model we would also like to push the program of quantization by finding an actual prequantum line bundle for $(\mathcal{M}_g, \omega)$. In [43] Wolpert obtains a line bundle on $\overline{\mathcal{M}}_g$ with curvature ω_{WP}. In future work we would like to understand this line bundle and its properties; in particular, we would like to explicitly compute the cohomologies that would allow us to determine the dimensionality of the phase space in the quantum regime. Finally, and perhaps most interestingly, we would like to look at extending this work to quantize the moduli spaces of surfaces with boundary, in hopes of making more explicit contact with the entropy of the BTZ black hole.

Bibliography

[1] Staruszkiewicz, A. “Gravitation theory in three-dimensional space.” Acta Physica Polonica **6** (1963) 735-740

[2] Achúcarro, A. and Townsend, P.K. “A Chern-Simons action for three-dimensional anti-de Sitter supergravity theories.” Phys. Lett. B **180** (1986) 89-92

[3] Witten, Edward. “(2+1) dimensional gravity as an exactly solvable system.” Nuclear Physics B **311** (1988) 46

[4] Strominger, Andrew. “Black Hole Entropy from Near Horizon Microstates.” arXiv:hep-th/9712251v3

[5] Ryu, S. and Takayanagi, T. “Holographic Derivation of Entanglement Entropy from AdS/CFT.” arXiv:hep-th/0603001

[6] Wolpert, Scott. “On the homology of the moduli space of stable curves.” Annals of Mathematics **118** (1983) 491

[7] Carroll, Sean. *Spacetime and Geometry: And Introduction to General Relativity.* (Addison Wesley, 2004)

[8] Carlip, Steven. “Quantum Gravity in 2+1 Dimensions: The Case of a Closed Universe.” arXiv:gr-qc/0409039v2

[9] Carlip, Steven. *Quantum Gravity in 2+1 Dimensions.* (Cambridge University Press, 1998)

[10] Arnowitt, R., Deser, S. and Misner, C. W. “The dynamics of general relativity” in *Gravitation: an introduction to current research,* ed. L. Witten (Wiley, 1962)

[11] Moncrief, Vincent. “Reduction of the Einstein Equations in 2+1 dimensions to a Hamiltonian system over Teichmüller space.” J. Math. Phys. **30** (1989) 2907

[12] York, James W. “Role of Conformal Three-Geometry in the Dynamics of Gravitation.” Phys. Rev. Lett. **28** (1972) 1082

[13] Puzio, Raymond S. “On the square root of the Laplace-Beltrami operator as a Hamiltonian.” Class. Quant. Grav. **11** (1994) 609.

[14] Moncrief, Vincent. “How solvable is (2+1)dimensional Einstein gravity?” J. Math. Phys. **31** (1990) 2978

[15] Killingback, T. P. ”Quantization of SL(2,R) Chern-Simons Theory.” Commun. Math. Phys. **145** (1992) 1-16

[16] Witten, Edward. “Quantum Field Theory and the Jones Polynomial.” Commun. Math. Phys. **121** (1989) 351-399

[17] Axelrod, S., Della Pietra, S. and Witten, E. “Geometric Quantization of Chern-Simons Gauge Theory.” J. Differential Geometry **33** (1991) 787

[18] Witten, Edward. “Three-Dimensional Gravity Reconsidered.” arXiv:0706.3359v1

[19] Hartle, J. B. and Hawking, S. W. “Wave function of the Universe.” Phys. Rev. D 28 (1983) 2960

[20] Hitoshi Murayama. “221A Lecture Notes: Landau Levels.” hitoshi.berkeley.edu/221a/landau.pdf

[21] Hirschfield, Peter J. “Charged Particle in a Magnetic FIeld.” http://www.phys.ufl.edu/∼pjh/teaching/phy4605/notes/landau.pdf

[22] Blau, Matthias. “Symplectic Geometry and Geometric Quantization.” Lecture notes (1992) www.blau.itp.unibe.ch/Lecturenotes.html

[23] Hassan, Syed Asif and Mainiero, Thomas. “Geometric Quantization.” http://ugcs.caltech.edu/∼mainiero/articles/geom_quant_mainiero_hassan.pdf

[24] Griffiths, Phillip and Harris, Joseph. *Principles of Algebraic Geometry.* (Wiley, 1994)

[25] Goldman, William M. “The Symplectic Nature of Fundamental Groups of Surfaces.” Adv. Math. **54** (1984) 200-225

[26] Witten, Edward. “On Quantum Gauge Theories in Two Dimensions.” Commun. Math. Phys. **141** (1991) 153-209

[27] Harer, John. “The Cohomology of the Moduli Space of Curves.” *Lecture Notes in Mathematics.* **1337**, Springer-Verlag, 138-221

[28] Do, Norman. “Moduli spaces of hyperbolic surfaces and their Weil-Petersson volumes.” arXiv:1103.4674v1

[29] Wolpert, Scott. “On the Weil-Petersson geometry of the moduli space of curves.” Amer. J. Math. **107** (1985) 969-997

[30] Dehn, M. “Die Gruppe der Abbildungklassen.” Acta Math. **69** (1938) 135-206

[31] Mirzakhani, Maryam.“Weil-Petersson volumes and intersection theory on the moduli space of curves.” J. Amer. Math. Soc. **20** (2007), 1-23 (electronic)

[32] Witten, Edward. “Two dimensional gravity and the intersection theory on moduli space.” Surveys in Diff. Geom. **1** (1991) 243-310

[33] Kontsevich, Maxim. “Intersection Theory of the Moduli Space of Curves and the Matrix Airy Function.” Commun. Math. Phys. **147** (1992) 1-23

[34] Zograf, P. “On the large genus asymptotics of Weil-Petersson volumes.” arXiv:0812.0544v1

[35] Mirzakhani, Maryam. “Growth of Weil-Petersson volumes and random hyperbolic surfaces of large genus.” arXiv:1012.2167v1

[36] Wolpert, Scott. “On the Kähler form of the moduli space of once punctured tori.” Comment. Math. Helvetici **58** (1983) 246

[37] Mirzakhani, Maryam. “Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces.” Invent. Math. **167** (2007) 179-222

[38] McShane, G. “A remarkable identity for lengths of curves.” Ph.D. thesis, University of Warwick (1991)

[39] Harer, John. “The second homology group of the mapping class group of an orientable suface.” Invent. Math. **72** (1982) 23-86

[40] Zwiebach, Barton. *A First Course in String Theory.* (Cambridge University Press, 2009)

[41] Witten, Edward. “Topology-changing amplitudes in $2 + 1$ dimensional gravity.” Nuclear Physics B **323** (1989) 113-140

[42] Bañados, M., Henneaux, M., Teitelboim, C. and Zanelli, J. “Geometry of the 2+1 Black Hole.” Phys. Rev. D **48** (1993) 1506-1525

[43] Wolpert, Scott. “On obtaining a positive line bundle from the Weil-Petersson class.” Amer. J. Math. **107** (1985) 1485-1507